Solutions to In-Class Problems Week 2, Mon.

Problem 1. Generalize the proof from lecture (reproduced below) that $\sqrt{2}$ is irrational, for example, how about $\sqrt[3]{2}$? Remember that an irrational number is a number that cannot be expressed as a ratio of two integers.

Theorem. $\sqrt{2}$ *is an irrational number.*

Proof. The proof is by contradiction. Assume for purpose of contradiction that $\sqrt{2}$ is rational.

Then we can write $\sqrt{2} = m/n$ where m and n are integers and the fraction is in lowest terms. Squaring both sides gives $2 = m^2/n^2$, so $2n^2 = m^2$. This implies that m^2 is even, and hence that m is even; that is, m is a multiple of 2. But that means m^2 is actually a multiple of 4, say $m^2 = 4k$.

Now we have $2n^2 = m^2 = 4k$, so $n^2 = 2k$. So n^2 is even, and hence n is even. But since m and n are both even, the fraction m/n is not in lowest terms, a contradiction. $\qquad\square$

Solution. We prove that for any $n > 1$, $\sqrt[n]{2}$ is irrational by contradiction.

Assume that $\sqrt[n]{2}$ is rational. Under this assumption, there exist integers a and b with $\sqrt[n]{2} = a/b$, where a and b have no common factors (so that the fraction a/b is in lowest terms). Now we prove that a and b are both even, which contradicts the existence of a, b.

$$
\begin{aligned}
\sqrt[n]{2} &= \frac{a}{b} \\
2 &= \frac{a^n}{b^n} \\
2b^n &= a^n.
\end{aligned}
$$

The lefthand side of the last equation is even, so a^n is even. This implies that a is even as well (see below for justification).

In particular, $a = 2c$ for some integer c. Thus,

$$
\begin{aligned}
2b^n &= (2c)^n = 2^n c^n, \\
b^n &= 2^{n-1} c^n.
\end{aligned}
$$

Since $n - 1 > 0$, the righthand side of the last equation is an even number, so b^n is even. But this implies that b must be even as well, contradicting the fact that m/n is in lowest terms.

Now we justify the claim that if a^n is even, so is a.

There is a simple proof by contradiction: suppose a was not even, *i.e.*, a is odd. It's a familiar (and easily verified) fact that the product of *any* number of odd numbers is odd, so a^n would also be odd, contradicting the fact that a^n is even.

But more generally for *any* integers $m, k > 0$, if m^k is divisible by a prime number, p, then m must be divisible by p. This follows from the factorization of an integer into primes (which we'll discuss further in a coming lecture): the primes in the factorization of m^k are precisely the primes in the factorization of m repeated k times, so if there is a p in the factorization of m^k it must be one of k copies of a p in the factorization of m.

[Optional]

Here's a somewhat broader generalization of the proof that $\sqrt{2}$ is irrational.

Lemma. *Let the coefficients of the polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + x^n$ be integers. Then any real root of the polynomial is either integral or irrational.*

Notice that this Lemma directly implies that $\sqrt{2}$ is irrational: $x^2 - 2$ has no integer root because 2 is not a perfect square, so the real roots, namely, $\pm\sqrt{2}$, must be irrational. Similarly, if $m \in \mathbb{N}$ is not the kth power of an integer, then $x^k - m$ has no integer roots, so $\sqrt[k]{m}$ will be irrational.

Proof. Let $r \in \mathbb{R}$ be a root of the polynomial, that is,

$$a_0 + a_1 r + a_2 r^2 + \cdots + a_{n-1}r^{n-1} + r^n = 0.$$

There are three cases for r: it is either integral, irrational, or the ratio of two integers with no common divisors where the denominator is greater than one. We want to eliminate the last case.

So assume to the contrary that $r = s/t$ for integers s and t which have no common divisors and such that $t > 1$. Now t must have a prime divisor, p.

Substituting s/t for r and multiplying both sides of the above equation by t^n yields:

$$a_0 t^n + a_1 s t^{n-1} + a_2 s^2 t^{n-2} + \cdots + a_{n-1}s^{n-1}t + s^n \quad = \quad 0, \tag{1}$$
$$a_0 t^n + a_1 s t^{n-1} + a_2 s^2 t^{n-2} + \ldots + s^{n-1}t \quad = \quad -s^n. \tag{2}$$

Then p divides each term of the lefthand side of equation (2), and therefore also divides the righthand side, viz., $p \mid -s^n$. But this means that p must also divide s. So p is a common divisor of s and t, contradicting our choice of s and t.

$\square$

$\blacksquare$

Problem 2. [1] A set of propositions is *consistent* if there is a possible situation in which they are all true. This problem examines whether the following specifications are consistent.

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[1]From Rosen, 5th edition, Exercise 1.1.36

1. If the file system is not locked, then

    (a) new messages will be queued.

    (b) new messages will be sent to the messages buffer.

    (c) the system is functioning normally, and conversely, if the system is functioning normally, then the file system is not locked.

2. If new messages are not queued, then they will be sent to the messages buffer.

3. New messages will not be sent to the message buffer.

(a) Begin by translating the parts of the specification into propositional formulas using four propositional variables:

$$
\begin{aligned}
L &::= \text{ file system locked,} \\
Q &::= \text{ new messages are queued,} \\
B &::= \text{ new messages are sent to the message buffer,} \\
N &::= \text{ system functioning normally.}
\end{aligned}
$$

Solution. The translations of the specifications are:

$$
\begin{array}{lr}
\neg L \longrightarrow Q & \text{(Spec. 1.(a))} \\
\neg L \longrightarrow B & \text{(Spec. 1.(b))} \\
\neg L \longleftrightarrow N & \text{(Spec. 1.(c))} \\
\neg Q \longrightarrow B & \text{(Spec. 2.)} \\
\neg B & \text{(Spec. 3.)}
\end{array}
$$

∎

(b) To be precise, the specification is consistent if there is a single choice of truth values for the variables L, Q, B, N, so that every one the propositional formulas from the previous part are true. Explain how to use a *truth table* to determine whether the specification is consistent.

Solution. We can construct a truth table with sixteen lines—one for each way of assigning truth values to the four variables L, N, Q, and B. For each line, we could record the truth values of these five statements above.

If all five statements are true for some assignment of truth values to the variables, then the system is consistent. If for every one of the sixteen possible truth assignments, at least one of the five statements is false, then the system is inconsistent. Carrying out the calculation shows that there is a unique assignment of True/False values to L, N, Q, and B (see below) that satisfies all the specifications. ∎

(c) Use *simple reasoning by cases* to find a truth assignment that confirms that this system specification is consistent. Explain why there is only one such assignment.

Solution. We can avoid the full truthtable calculation if we reason by cases.

Case 1 (B is True): Then the last formula, (Spec. 3.), is false, and the whole specification is false.

Case 2 (B is False): Now (Spec. 2.) and (Spec. 1.(b)) can be true only if Q and L are true. Since L is true, (Spec. 1.(c)) can be true only if N is false. Thus, we have deduced that in order to be consistent, we must have

$$
\begin{aligned}
L &= \text{True} \\
N &= \text{False} \\
Q &= \text{True} \\
B &= \text{False.}
\end{aligned}
$$

From the way this assignment was constructed, we know it ensures that formulas from (Spec. 1.(b)) on are true. So all that remains is to check formula (Spec. 1.(a)), and indeed it is also true under this assignment.

So the system is consistent, and this is the only assignment that will satisfy it. ∎

Problem 3. If we raise an irrational number to an irrational power, can the result be rational? Show that it can by considering $\sqrt{2}^{\sqrt{2}}$ and arguing by cases.

Solution. We want to find irrational numbers a, b such that a^b is rational. We argue by cases.

Case 1: [$\sqrt{2}^{\sqrt{2}}$ is rational]. Let $a = b = \sqrt{2}$. a and b are irrational since $\sqrt{2}$ is irrational as we know. Also, a^b is rational by case hypothesis. So we have found the required a and b in this case.

Case 2: [$\sqrt{2}^{\sqrt{2}}$ is irrational]. Let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Then a is irrational by case hypothesis, we know b is irrational, and

$$
a^b = \left(\sqrt{2}^{\sqrt{2}} \right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2,
$$

which is rational. So we have found the required a and b in this case also.

So in any case, there will be irrational a, b such that a^b is rational. ∎