Solutions to In-Class Problems Week 15, Wed.

Gamblers Ruin

A gambler aims to gamble until he reaches a *goal* of T dollars or until he runs out of money, in which case he is said to be "ruined." He gambles by making a sequence of 1 dollar bets. If he wins an individual bet, his stake increases by one dollar. If he loses, his stake decreases by one dollar. In each bet, he wins with probability $p > 0$ and loses with probability $q ::= 1 - p > 0$. He is an overall *winner* if he reaches his goal and is an overall *loser* if he gets ruined.

In a *fair* game, $p = q = 1/2$. The gambler is more likely to win if $p > 1/2$ and less likely to win if $p < 1/2$.

With T and p fixed, the gambler's probability of winning will depend on how much money he starts with. Let w_n be the probability that he is a winner when his initial stake in n dollars.

Problem 1. **(a)** What are w_0 and w_T?

Solution. $w_0 = 0$ and $w_T = 1$. ■

(b) Note that w_n satisfies a linear recurrence

$$w_{n+1} = aw_n + bw_{n-1} \tag{1}$$

for some constants a, b and $0 < n < T$. Write simple expressions for a and b in terms of p.

Solution. By Total Probability

$$\begin{aligned} w_n &= \Pr\{\text{win game} \mid \text{win the first bet}\} \Pr\{\text{win the first bet}\} + \\ &\quad \Pr\{\text{win game} \mid \text{lose the first bet}\} \Pr\{\text{lose the first bet}\} \qquad (2)\\ &= pw_{n+1} + q \Pr\{w_{n-1}\}, \qquad \text{so} \\ pw_{n+1} &= w_n - qw_{n-1} \\ w_{n+1} &= \frac{w_n}{p} - \frac{qw_{n-1}}{p}. \qquad (3) \end{aligned}$$

So

$$a = \frac{1}{p}, \qquad b = -\frac{q}{p}.$$

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(c) For $n > T$, let w_n be defined by the recurrence (1), and let $g(x) ::= \sum_{n=1}^{\infty} w_n x^n$ be the generating function for the sequence $w_0, w_1, \ldots$. Verify that

$$g(x) = \frac{w_1 x}{(1-x)(1-\frac{q}{p}x)}. \tag{4}$$

Solution.

$$\begin{array}{rcllll} g(x) & = & w_0 \;+\; w_1x & +\; w_2x^2 & +\; w_3x^3 & + \cdots \\ xg(x)/p & = & \quad\;\; w_0x/p & +\; w_1x^2/p & +\; w_2x^3/p & + \cdots \\ (q/p)x^2g(x) & = & & \;\;(q/p)w_0x^2 & +\; (q/p)w_1x^3 & + \cdots \end{array}$$

so

$$g(x) - \left(\frac{xg(x)}{p} - \frac{qx^2g(x)}{p}\right) = w_0 + w_1x - w_0x/p = w_1x,$$

$$g(x)\left(1 - \frac{x}{p} + \frac{qx^2}{p}\right) = w_1x. \tag{5}$$

But

$$1 - \frac{x}{p} + \frac{qx^2}{p} = (1-x)(1-\frac{q}{p}x) \tag{6}$$

Combining (6) and (5) yields (4). ■

(d) Conclude that in an unfair game

$$w_n = c + d\left(\frac{q}{p}\right)^n \tag{7}$$

for some constants c, d.

Solution. In an unfair game $p/q \neq 1$, so from (4), we know that there will be c, d such that

$$g(x) = \frac{c}{1-x} + \frac{d}{1-\frac{q}{p}x} \tag{8}$$

so w_n will be the corresponding combination of the coefficients of x^n in $1/(1-x)$ and $1/(1-(q/p)x)$, namely, (7). ■

(e) Show that in an unfair game,

$$w_n = \frac{(q/p)^n - 1}{(q/p)^T - 1}.$$

Solution. Given (4), we want c, d such that

$$\frac{w_1 x}{(1-x)(1-\frac{q}{p}x)} = \frac{c}{1-x} + \frac{d}{1-\frac{q}{p}x}.$$

So c, d satisfy

$$w_1 x = c(1 - \frac{q}{p}x) + d(1-x).$$

Letting $x = 1$ gives

$$c = \frac{w_1}{1 - q/p}.$$

Letting $x = p/q$ gives

$$d = \frac{pw_1/q}{1-p/q} = \frac{w_1}{q/p - 1} = -c.$$

So plugging into (7) gives

$$w_n = \frac{w_1}{q/p-1}\left(\left(\frac{q}{p}\right)^n - 1\right). \tag{9}$$

Now we can solve for w_1, by letting $n = T$ in (9):

$$1 = w_T = \frac{w_1}{q/p-1}\left(\left(\frac{q}{p}\right)^T - 1\right)$$

so

$$w_1 = \frac{(q/p-1)}{(q/p)^T - 1}.$$

Combining this with (9) yields

$$w_n = \frac{((q/p)^n - 1)}{(q/p)^T - 1}.$$

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(f) Verify that if $0 < a < b$, then

$$\frac{a}{b} < \frac{a+1}{b+1}.$$

Conclude that if $p < 1/2$, then

$$w_n < \left(\frac{p}{q}\right)^{T-n}.$$

Solution.

$$\frac{a}{b} = \frac{a(1+1/b)}{b(1+1/b)} = \frac{a+a/b}{b+1} < \frac{a+1}{b+1}.$$

So from the previous part, we have

$$w_n = \frac{(q/p)^n - 1}{(q/p)^T - 1} < \frac{(q/p)^n}{(q/p)^T} = \left(\frac{q}{p}\right)^{n-T} = \left(\frac{p}{q}\right)^{T-n}.$$

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Problem 2. Show that in a fair game,

$$w_n = \frac{w}{T}.$$

Hint: Use equation (4) again.

Solution. This time $p = q = 1/2$ so from (4),

$$g(x) = \frac{w_1 x}{(1-x)^2}.$$

Now we need a, b such that

$$\frac{w_1 x}{(1-x)^2} = \frac{a}{1-x} + \frac{b}{(1-x)^2}, \tag{10}$$

so we will have

$$w_n = a + b(n+1).$$

Solving for a, b, we have from (10)

$$w_1 x = a(1-x) + b.$$

Letting $x = 0$ yields $a = -b$ and $x = 1$ yields $b = w_1$, so

$$w_n = -w_1 + w_1(n+1) = w_1 n.$$

Also,

$$1 = w_T = w_1 T$$

so

$$w_1 = \frac{1}{T}$$

and hence

$$w_n = \frac{n}{T}.$$

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Problem 3. Now suppose $T = \infty$, that is, the gambler keeps playing until he is ruined. (Now there may be a positive probability that he actually plays forever.) Let r be the probability that starting with $n > 0$ dollars, the gambler's stake ever gets reduced to $n - 1$.

(a) Explain why

$$r = q + pr^2.$$

Solution. By Total Probability

$$\begin{aligned} r &= \Pr\{\text{ever down \$1} \mid \text{lose the first bet}\} \Pr\{\text{lose the first bet}\} + \\ &\qquad \Pr\{\text{ever down \$1} \mid \text{win the first bet}\} \Pr\{\text{win the first bet}\} \\ &= q + p \Pr\{\text{ever down \$1} \mid \text{win the first bet}\} \end{aligned}$$

But

$$\begin{aligned} &\Pr\{\text{ever down \$1} \mid \text{win the first bet}\} \\ &= \Pr\{\text{ever down \$2}\} \\ &= \Pr\{\text{being down the first \$1}\} \Pr\{\text{being down another \$1}\} \\ &= r^2. \end{aligned}$$

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(b) Conclude that if $p \leq 1/2$, then $r = 1$.

Solution. $pr^2 - r + q$ has roots q/p and 1. So $r = 1$ or $r = q/p$. But $1 \leq r$, which implies $r = 1$ when $q/p \geq 1$, that is, when $p \leq 1/2$.

In fact $r = q/p$ when $q/p < 1$, namely, when $p > 1/2$, but this requires an additional argument that we omit. ■

(c) Conclude that even in a fair game, the gambler is sure to get ruined *no matter how much money he starts with*!

Solution. The gambler gets ruined starting with initial stake $n = 1$ precisely if his initial stake goes down by 1 dollar, so his probability of ruin is r, which equals 1 in the fair case.

The recurrence (1) will also hold in this $T = \infty$ case if we interpret w_n as the probability of *not* being ruined, that is, the gambler wins if he can gamble forever. So w_1 is the probability he is *not* getting ruined starting with a 1 dollar stake, that is $w_1 = 1 - r = 0$. Since $w_0 = 0 = w_1$, the recurrence implies that $w_n = 0$ for all $n \geq 0$. ■

(d) Let t be the expected time for the gambler's stake to go down by 1 dollar. Verify that

$$t = q + p(1 + 2t).$$

Conclude that starting with a 1 dollar stake in a fair game, the gambler can expect to play forever!

Solution. By Total Expectation

$$\begin{aligned} t &= \text{E}\left[\text{\#steps to be down \$1} \mid \text{lose the first bet}\right] \Pr\left\{\text{lose the first bet}\right\} + \\ &\qquad \text{E}\left[\text{\#steps to be down \$1} \mid \text{win the first bet}\right] \Pr\left\{\text{win the first bet}\right\} \\ &= q + p\,\text{E}\left[1 + \text{\#steps to be down \$1} \mid \text{win the first bet}\right]. \end{aligned}$$

But

$$\begin{aligned} &\text{E}\left[\text{\#steps to be down \$1} \mid \text{win the first bet}\right] \\ &= \text{E}\left[\text{\#steps to be down \$2}\right] \\ &= \text{E}\left[\text{\#steps to be down the first \$1}\right] + \text{E}\left[\text{\#steps to be down another \$1}\right] \\ &= 2t. \end{aligned}$$

This implies the required formula $t = q + p(1 + 2t)$. If $p = 1/2$ we conclude that $t = 1 + t$, which means t must be infinite. ■