In-Class Problems Week 15, Wed.

Gamblers Ruin

A gambler aims to gamble until he reaches a *goal* of T dollars or until he runs out of money, in which case he is said to be "ruined." He gambles by making a sequence of 1 dollar bets. If he wins an individual bet, his stake increases by one dollar. If he loses, his stake decreases by one dollar. In each bet, he wins with probability $p > 0$ and loses with probability $q ::= 1 - p > 0$. He is an overall *winner* if he reaches his goal and is an overall *loser* if he gets ruined.

In a *fair* game, $p = q = 1/2$. The gambler is more likely to win if $p > 1/2$ and less likely to win if $p < 1/2$.

With T and p fixed, the gambler's probability of winning will depend on how much money he starts with. Let w_n be the probability that he is a winner when his initial stake in n dollars.

Problem 1. (a) What are w_0 and w_T?

(b) Note that w_n satisfies a linear recurrence

$$w_{n+1} = aw_n + bw_{n-1} \tag{1}$$

for some constants a, b and $0 < n < T$. Write simple expressions for a and b in terms of p.

(c) For $n > T$, let w_n be defined by the recurrence (1), and let $g(x) ::= \sum_{n=1}^{\infty} w_n x^n$ be the generating function for the sequence $w_0, w_1, \ldots$. Verify that

$$g(x) = \frac{w_1 x}{(1-x)(1-\frac{q}{p}x)}. \tag{2}$$

(d) Conclude that in an unfair game

$$w_n = c + d\left(\frac{q}{p}\right)^n \tag{3}$$

for some constants c, d.

(e) Show that in an unfair game,

$$w_n = \frac{(q/p)^n - 1}{(q/p)^T - 1}.$$

(f) Verify that if $0 < a < b$, then

$$\frac{a}{b} < \frac{a+1}{b+1}.$$

Conclude that if $p < 1/2$, then

$$w_n < \left(\frac{p}{q}\right)^{T-n}.$$

Problem 2. Show that in a fair game,

$$w_n = \frac{w}{T}.$$

Problem 3. Now suppose $T = \infty$, that is, the gambler keeps playing until he is ruined. (Now there may be a positive probability that he actually plays forever.) Let r be the probability that starting with $n > 0$ dollars, the gambler's stake ever gets reduced to $n - 1$.

(a) Explain why

$$r = q + pr^2.$$

(b) Conclude that if $p \leq 1/2$, then $r = 1$.

(c) Conclude that even in a fair game, the gambler is sure to get ruined *no matter how much money he starts with*!

(d) Let t be the expected time for the gambler's stake to go down by 1 dollar. Verify that

$$t = q + p(1 + 2t).$$

Conclude that starting with a 1 dollar stake in a fair game, the gambler can expect to play forever!