

In-Class Problems Week 14, Wed.

Problem 1. Suppose you have learned that the average graduating MIT student's total number of credits is 200.

(a) Knowing only this average, use Markov's inequality to find a best possible upper bound for the fraction of MIT students graduating with at least 235 credits.¹

(b) Demonstrate that this is a best possible bound by giving a distribution for which this bound holds with equality.

(c) Suppose you are now told that no student can graduate with fewer than 170 units. How does this allow you to improve your previous bound? As before, show that this is the best possible bound.

(d) Now suppose you *further* learn that the standard deviation of the total credits per graduating student is 7. What is the Chebyshev bound on the fraction of students who can graduate with at least 235 credits?

Problem 2. (a) Show that Markov's Theorem only applies to nonnegative random variables. That is, give an example of a random variable to which Markov's Theorem gives a *wrong* answer.

(b) Suppose R is a random variable that is always at least -10 and has expectation 0 . Since R may be negative, Markov's theorem does not apply directly. Still, use Markov's theorem to show that the probability that R is ≥ 5 is at most $2/3$.

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¹Ignore the fact that there are practical limits to the amount of time a student can stay at MIT and remain sane; That is, assume that there is no bound on the number of credits a student may earn.

Problem 3. There are n people at a circular table in a Chinese restaurant. On the table, there are n different appetizers arranged on a big Lazy Susan. Each person starts munching on the appetizer directly in front of him or her. Then someone spins the Lazy Susan so that everyone is faced with a random appetizer. In class, we saw that the expected number of people that end up with the appetizer that they had originally is 1.

Let X_i be the indicator variable for the i th person getting their own appetizer back. Let S_n be the total number of people who get their own appetizer back, so $S_n = \sum_{i=1}^n X_i$.

- (a) What is $E[X_i^2]$?
- (b) For $i \neq j$, what is $E[X_i X_j]$?
- (c) What is $E[S_n^2]$?
- (d) What is $\text{Var}[S_n]$?
- (e) Discuss the accuracy of the Chebyshev Bound on the probability that S_n is distance x from its expectation as x ranges over integers between 1 and n .

Problem 4. For any random variable, R , with $E[R] = \mu$ and $\text{Var}[R] = v$, the Chebyshev Bound says that for any real number $x > 0$,

$$\Pr\{|R - \mu| \geq x\} \leq \frac{v}{x^2}.$$

Show that for any real number, μ , and real numbers $v, x > 0$, there is an R for which the Chebyshev Bound is tight, that is,

$$\Pr\{|R| \geq x\} = \frac{v}{x^2}. \quad (1)$$

Hint: Assume $\mu = 0$ and let R be three valued with values 0, $-x$, and x .

Problem 5. The covariance, $\text{Cov}[X, Y]$, of two random variables, X and Y , is defined to be $E[XY] - E[X]E[Y]$. Note that if two random variables are independent, then their covariance is zero.

- (a) Give an example to show that having $\text{Cov}[X, Y] = 0$ does not necessarily mean that X and Y are independent.
- (b) Let X_1, \dots, X_n be random variables. Prove that

$$\text{Var}[X_1 + \dots + X_n] = \sum_{i=1}^n \text{Var}[X_i] + 2 \sum_{i < j} \text{Cov}[X_i, X_j]$$