In-Class Problems Week 14, Mon.

Problem 1. Here are seven propositions:

$$
\begin{array}{rcrcr}
x_1 & \lor & x_3 & \lor & \neg x_7 \\
\neg x_5 & \lor & x_6 & \lor & x_7 \\
x_2 & \lor & \neg x_4 & \lor & x_6 \\
\neg x_4 & \lor & x_5 & \lor & \neg x_7 \\
x_3 & \lor & \neg x_5 & \lor & \neg x_8 \\
x_9 & \lor & \neg x_8 & \lor & x_2 \\
\neg x_3 & \lor & x_9 & \lor & x_4
\end{array}
$$

Note that:

1. Each proposition is the OR of three terms of the form x_i or the form $\neg x_i$.
2. The variables in the three terms in each proposition are all different.

Suppose that we assign true/false values to the variables $x_1, \ldots, x_9$ independently and with equal probability.

(a) What is the probability that a single proposition is true?

(b) What is the expected number of true propositions?

(c) Use your answer to prove that there exists an assignment to the variables that makes *all* of the propositions true.

Problem 2. Final exams in 6.042 are graded according to a rigorous procedure:

- With probability $4/7$ the exam is graded by a *recitation instructor*, with probability $2/7$ it is graded by a *lecturer*, and with probability $1/7$, it is accidentally dropped behind the radiator and arbitrarily given a score of 84.
- *Recitation instructors* score an exam by scoring each problem individually and then taking the sum.
  - There are ten true/false questions worth 2 points each. For each, full credit is given with probability $3/4$, and no credit is given with probability $1/4$.
  - There are four questions worth 15 points each. For each, the score is determined by rolling two fair dice, summing the results, and adding 3.
  - The single 20 point question is awarded either 12 or 18 points with equal probability.
- *Lecturers* score an exam by rolling a fair die twice, multiplying the results, and then adding a "general impression" score.
  - With probability $4/10$, the general impression score is 40.
  - With probability $3/10$, the general impression score is 50.
  - With probability $3/10$, the general impression score is 60.

Assume all random choices during the grading process are mutually independent.

(a) What is the expected score on an exam graded by a recitation instructor?

(b) What is the expected score on an exam graded by a lecturer?

(c) What is the expected score on a 6.042 exam?

Problem 3. The number of squares that a piece advances in one turn of the game Monopoly is determined as follows:

- Roll two dice, take the sum of the numbers that come up, and advance that number of squares.
- If you roll *doubles* (that is, the same number comes up on both dice), then you roll a second time, take the sum, and advance that number of additional squares.
- If you roll doubles a second time, then you roll a third time, take the sum, and advance that number of additional squares.

- However, as a special case, if you roll doubles a third time, then you go to jail. Regard this as advancing zero squares overall for the turn.

(a) What is the expected sum of two dice, given that the same number comes up on both?

(b) What is the expected sum of two dice, given that different numbers come up? (Use your previous answer and the Total Expectation Theorem.)

(c) To simplify the analysis, suppose that we always roll the dice three times, but may ignore the second or third rolls if we didn't previously get doubles. Let the random variable X_i be the sum of the dice on the i-th roll, and let E_i be the event that the i-th roll is doubles. Write the expected number of squares a piece advances in these terms.

(d) What is the expected number of squares that a piece advances in Monopoly?