Solutions to In-Class Problems Week 11, Wed.

Problem 1. Define the function $f : \mathbb{N} \to \mathbb{N}$ recursively by the rules

$$\begin{aligned} f(0) &= 1, \\ f(1) &= 6, \\ f(n) &= 2f(n-1) + 3f(n-2) + 4 \qquad \text{for } n \geq 2. \end{aligned}$$

(a) Find a closed form for the generating function

$$G(x) ::= f(0) + f(1)x + f(2)x^2 + \cdots + f(n)x^n + \cdots.$$

Solution.

$$\begin{array}{rcllllll} G(x) &=& f(0) &+ f(1)x &+ f(2)x^2 &+\cdots &+ & f(n)x^n + \cdots \\ 2xG(x) &=& & 2f(0)x &+ 2f(1)x^2 &+\cdots &+ & 2f(n-1)x^n + \cdots \\ 3x^2G(x) &=& & & 3f(0)x^2 &+\cdots &+ & 3f(n-2)x^n + \cdots \\ 4/(1-x) &=& 4 &+ 4x & 4x^2 &+\cdots &+ & 4x^n + \cdots \end{array}$$

Therefore,

$$\begin{aligned} G(x) &= 2xG(x) + 3x^2G(x) + \frac{4}{1-x} + (f(0)-4) + (f(1)-2f(0)-4)x \\ &= 2xG(x) + 3x^2G(x) + \frac{4}{1-x} + (1-4) + (6-2-4)x \\ &= 2xG(x) + 3x^2G(x) + \frac{4}{1-x} - 3, \end{aligned}$$

It follows that

$$G(x)(1-2x-3x^2) = \frac{4}{1-x} - 3,$$

and hence

$$\begin{aligned} G(x) &= \frac{\frac{4}{1-x} - 3}{(1+x)(1-3x)} \\ &= \frac{4}{(1-x)(1+x)(1-3x)} - \frac{3}{(1+x)(1-3x)} \\ &= \frac{4-3(1-x)}{(1-x)(1+x)(1-3x)} \\ &= \frac{3x+1}{(1-x)(1+x)(1-3x)}. \end{aligned} \tag{1}$$

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(b) Find a closed form for $f(n)$. *Hint:* Find numbers a, b, c, d, e, g such that

$$G(x) = \frac{a}{1+dx} + \frac{b}{1+ex} + \frac{c}{1+gx}.$$

Solution. From (1) and the method of partial fractions, we conclude that $d, e, g = -1, 1, -3$, respectively. So we want a, b, c such that

$$\frac{3x+1}{(1-x)(1+x)(1-3x)} = \frac{a}{1-x} + \frac{b}{1+x} + \frac{c}{1-3x} \quad (2)$$

$$3x+1 = a(1+x)(1-3x) + b(1-x)(1-3x) + c(1-x)(1+x). \quad (3)$$

Setting $x = 1$ in (3), we conclude that $4 = a \cdot 2 \cdot (-2)$, so

$$a = -1.$$

Setting $x = -1$ in (3), we conclude that $4 - 3 \cdot 2 = b \cdot 2 \cdot 4$, so

$$b = -\frac{1}{4}.$$

Setting $x = 1/3$ in (3), we conclude that $4 - 3(2/3) = c \cdot (2/3)(4/3)$, so

$$c = \frac{9}{4}.$$

So from (1) and (2), we have

$$G(x) = \frac{-1}{1-x} + \frac{1/4}{1+x} + \frac{9/4}{1-3x}.$$

Now the coefficient of x^n in $a/(1-x)$ is a, the coefficient in $b/(1+x)$ is $b(-1)^n$ and the coefficient in $c/(1-3x)$ is $c3^n$. For $n \geq 2$, the coefficient in $G(x)$ is the sum of these coefficients. So

$$f(n) = -1 + \frac{(-1)^n}{4} + \frac{9}{4}3^n = \frac{3^{n+2} + (-1)^n}{4} - 1.$$

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Appendix

Finding a Generating Function for Fibonacci Numbers

The Fibonacci numbers are defined by:

$$\begin{aligned} f_0 &::= 0 \\ f_1 &::= 1 \\ f_n &::= f_{n-1} + f_{n-2} \qquad \text{(for } n \geq 2) \end{aligned}$$

Let F be the generating function for the Fibonacci numbers, that is,

$$F(x) ::= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots$$

So we need to derive a generating function whose series has coefficients:

$$\langle 0,\ 1,\ f_1 + f_0,\ f_2 + f_1,\ f_3 + f_2,\ \ldots \rangle$$

Now we observe that

$$\begin{array}{rcccccccl} & \langle\ 0, & 1, & 0, & 0, & 0, & \ldots\ \rangle & \longleftrightarrow & x \\ & \langle\ 0, & f_0, & f_1, & f_2, & f_3, & \ldots\ \rangle & \longleftrightarrow & xF(x) \\ + & \langle\ 0, & 0, & f_0, & f_1, & f_2, & \ldots\ \rangle & \longleftrightarrow & x^2F(x) \\ \hline & \langle\ 0, & 1+f_0, & f_1+f_0, & f_2+f_1, & f_3+f_2, & \ldots\ \rangle & \longleftrightarrow & x + xF(x) + x^2F(x) \end{array}$$

This sequence is almost identical to the right sides of the Fibonacci equations. The one blemish is that the second term is $1 + f_0$ instead of simply 1. But since $f_0 = 0$, the second term is ok.

So we have

$$\begin{aligned} F(x) &= x + xF(x) + x^2F(x). \\ F(x) &= \frac{x}{1 - x - x^2}. \end{aligned} \tag{4}$$

Finding a Closed Form for the Coefficients

Now we expand the righthand side of (4) into partial fractions. To do this, we first factor the denominator

$$1 - x - x^2 = (1 - \alpha_1 x)(1 - \alpha_2 x)$$

where $\alpha_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\alpha_2 = \frac{1}{2}(1 - \sqrt{5})$ by the quadratic formula. Next, we find A_1 and A_2 which satisfy:

$$F(x) = \frac{x}{1 - x - x^2} = \frac{A_1}{1 - \alpha_1 x} + \frac{A_2}{1 - \alpha_2 x} \tag{5}$$

Now the coefficient of x^n in $F(x)$ will be A_1 times the coefficient of x^n in $1/(1 - \alpha_1 x)$ plus A_2 times the coefficient of x^n in $1/(1 - \alpha_2 x)$. The coefficients of these fractions will simply be the terms α_1^n and α_2^n because

$$\begin{aligned} \frac{1}{1 - \alpha_1 x} &= 1 + \alpha_1 x + \alpha_1^2 x^2 + \cdots \\ \frac{1}{1 - \alpha_2 x} &= 1 + \alpha_2 x + \alpha_2^2 x^2 + \cdots \end{aligned}$$

by the formula for geometric series.

So we just need to find find A_1 and A_2. We do this by plugging values of x into (5) to generate linear equations in A_1 and A_2. It helps to note that from (5), we have

$$x = A_1(1 - \alpha_2 x) + A_2(1 - \alpha_1 x),$$

so simple values to use are $x = 0$ and $x = 1/\alpha_2$. We can then find A_1 and A_2 by solving the linear equations. This gives:

$$\begin{aligned} A_1 &= \frac{1}{\alpha_1 - \alpha_2} = \frac{1}{\sqrt{5}} \\ A_2 &= -A_1 = -\frac{1}{\sqrt{5}} \end{aligned}$$

Substituting into (5) gives the partial fractions expansion of $F(x)$:

$$F(x) = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \alpha_1 x} - \frac{1}{1 - \alpha_2 x} \right).$$

So we conclude that the coefficient, f_n, of x^n in the series for $F(x)$ is

$$\begin{aligned} f_n &= \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right) \end{aligned}$$