In-Class Problems Week 11, Wed.

Problem 1. Define the function $f : \mathbb{N} \to \mathbb{N}$ recursively by the rules

$$\begin{aligned} f(0) &= 1, \\ f(1) &= 6, \\ f(n) &= 2f(n-1) + 3f(n-2) + 4 \qquad \text{for } n \geq 2. \end{aligned}$$

(a) Find a closed form for the generating function

$$G(x) ::= f(0) + f(1)x + f(2)x^2 + \cdots + f(n)x^n + \cdots .$$

(b) Find a closed form for $f(n)$. *Hint:* Find numbers a, b, c, d, e, g such that

$$G(x) = \frac{a}{1+dx} + \frac{b}{1+ex} + \frac{c}{1+gx}.$$

Appendix

Finding a Generating Function for Fibonacci Numbers

The Fibonacci numbers are defined by:

$$\begin{aligned} f_0 &::= 0 \\ f_1 &::= 1 \\ f_n &::= f_{n-1} + f_{n-2} \qquad (\text{for } n \geq 2) \end{aligned}$$

Let F be the generating function for the Fibonacci numbers, that is,

$$F(x) ::= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots$$

So we need to derive a generating function whose series has coefficients:

$$\langle 0,\ 1,\ f_1 + f_0,\ f_2 + f_1,\ f_3 + f_2,\ \ldots \rangle$$

Now we observe that

$$\begin{array}{rlllllll} & \langle\ 0, & 1, & 0, & 0, & 0, & \ldots\ \rangle & \longleftrightarrow\ x \\ & \langle\ 0, & f_0, & f_1, & f_2, & f_3, & \ldots\ \rangle & \longleftrightarrow\ xF(x) \\ + & \langle\ 0, & 0, & f_0, & f_1, & f_2, & \ldots\ \rangle & \longleftrightarrow\ x^2F(x) \\ \hline & \langle\ 0, & 1+f_0, & f_1+f_0, & f_2+f_1, & f_3+f_2, & \ldots\ \rangle & \longleftrightarrow\ x + xF(x) + x^2F(x) \end{array}$$

This sequence is almost identical to the right sides of the Fibonacci equations. The one blemish is that the second term is $1 + f_0$ instead of simply 1. But since $f_0 = 0$, the second term is ok.

So we have

$$\begin{aligned} F(x) &= x + xF(x) + x^2F(x). \\ F(x) &= \frac{x}{1-x-x^2}. \end{aligned} \tag{1}$$

Finding a Closed Form for the Coefficients

Now we expand the righthand side of (1) into partial fractions. To do this, we first factor the denominator

$$1 - x - x^2 = (1 - \alpha_1 x)(1 - \alpha_2 x)$$

where $\alpha_1 = \frac{1}{2}(1+\sqrt{5})$ and $\alpha_2 = \frac{1}{2}(1-\sqrt{5})$ by the quadratic formula. Next, we find A_1 and A_2 which satisfy:

$$F(x) = \frac{x}{1-x-x^2} = \frac{A_1}{1-\alpha_1 x} + \frac{A_2}{1-\alpha_2 x} \tag{2}$$

Now the coefficient of x^n in $F(x)$ will be A_1 times the coefficient of x^n in $1/(1-\alpha_1 x)$ plus A_2 times the coefficient of x^n in $1/(1-\alpha_2 x)$. The coefficients of these fractions will simply be the terms α_1^n and α_2^n because

$$\begin{aligned} \frac{1}{1-\alpha_1 x} &= 1 + \alpha_1 x + \alpha_1^2 x^2 + \cdots \\ \frac{1}{1-\alpha_2 x} &= 1 + \alpha_2 x + \alpha_2^2 x^2 + \cdots \end{aligned}$$

by the formula for geometric series.

So we just need to find find A_1 and A_2. We do this by plugging values of x into (2) to generate linear equations in A_1 and A_2. It helps to note that from (2), we have

$$x = A_1(1-\alpha_2 x) + A_2(1-\alpha_1 x),$$

so simple values to use are $x = 0$ and $x = 1/\alpha_2$. We can then find A_1 and A_2 by solving the linear equations. This gives:

$$\begin{aligned} A_1 &= \frac{1}{\alpha_1 - \alpha_2} = \frac{1}{\sqrt{5}} \\ A_2 &= \frac{-1}{\alpha_1 - \alpha_2} = -\frac{1}{\sqrt{5}} \end{aligned}$$

Substituting into (2) gives the partial fractions expansion of $F(x)$:

$$F(x) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha_1 x} - \frac{1}{1-\alpha_2 x}\right).$$

So we conclude that the coefficient, f_n, of x^n in the series for $F(x)$ is

$$\begin{aligned} f_n &= \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \end{aligned}$$