Solutions to In-Class Problems Week 11, Fri.

Problem 1. **(a)** Verify that

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n.$$

Hint: Use the fact that if $A(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$a_n = \frac{A^{(n)}(0)}{n!},$$

where $A^{(n)}$ is the nth derivative of A.

Solution.

$$\begin{aligned}
\frac{d\,(1-x)^{-k}}{dx} &= k\,(1-x)^{-(k+1)}. \\
\frac{d^2\,(1-x)^{-k}}{(dx)^2} &= \frac{d\,k\,(1-x)^{-(k+1)}}{dx} = (k+1)k\,(1-x)^{-(k+2)} \\
\frac{d^3\,(1-x)^{-k}}{(dx)^3} &= \frac{d\,(k+1)k\,(1-x)^{-(k+2)}}{dx} = (k+2)(k+1)k\,(1-x)^{-(k+3)} \\
&\vdots \\
\frac{d^n\,(1-x)^{-k}}{(dx)^n} &= (k+n-1)\cdots(k+2)(k+1)k\,(1-x)^{-(k+n)}.
\end{aligned}$$

Now suppose $(1-x)^{-k} = A(x)$. Then by the hint, we have

$$\begin{aligned}
a_n &= \frac{A^{(n)}(0)}{n!} \\
&= \frac{(k+n-1)\cdots(k+2)(k+1)k\,(1-0)^{-(k+n)}}{n!} \\
&= \frac{\frac{(n+k-1)!}{(k-1)!}\cdot 1}{n!} \\
&= \frac{(n+k-1)!}{(k-1)!n!} \\
&= \binom{n+k+1}{n}
\end{aligned}$$

■

(b) Let $S(x) ::= \sum_{k=1}^{\infty} k^2 x^k$. Explain why $S(x)/(1-x)$ is the generating function for the sums of squares. That is, the coefficient of x^n in the series for $S(x)/(1-x)$ is $\sum_{k=1}^{n} k^2$.

Solution.

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k \cdot 1\right) x^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k\right) x^n \tag{1}$$

by the convolution formula for the product of series. For $S(x)$, the coefficient of x^k is $a_k = k^2$, and

$$S(x)/(1-x) = S(x)\left(\sum_{n=0}^{\infty} x^n\right),$$

so (1) implies that the coefficient of x^n in $S(x)/(1-x)$ is the sum of the first n squares. ■

(c) Use the fact that

$$S(x) = \frac{x(1+x)}{(1-x)^3},$$

and the previous part to prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution. We have

$$\frac{S(x)}{1-x} = \frac{\frac{x(1+x)}{(1-x)^3}}{1-x} = \frac{x+x^2}{(1-x)^4}. \tag{2}$$

From part (a), the coefficient of x^n in the series expansion of $1/(1-x)^4$ is

$$\binom{n+3}{n} = \frac{(n+1)(n+2)(n+3)}{3!}.$$

But by (2),

$$\frac{S(x)}{1-x} = \frac{x}{(1-x)^4} + \frac{x^2}{(1-x)^4},$$

so the coefficient of x^n is the sum of the $(n-1)$st and $(n-2)$nd coefficients of $(1-x)^4$, namely,

$$\frac{n(n+1)(n+2)}{3!} + \frac{(n-1)n(n+1)}{3!} = \frac{n(n+1)(2n+1)}{6}.$$

■

(d) (Optional) How about a formula for the sum of cubes?

Solution. TBA ■

Problem 2. We are interested in generating functions for the number of different ways to compose a bag of n donuts subject to various restrictions. For each of the restrictions in (a)-(e) below, find a closed form for the corresponding generating function.

(a) All the donuts are chocolate and there are at least 3.

Solution.

$$\frac{x^3}{1-x}$$

■

(b) All the donuts are glazed and there are at most 2.

Solution.

$$1 + x + x^2$$

■

(c) All the donuts are coconut and there are exactly 2 or there are none.

Solution.

$$1 + x^2$$

■

(d) All the donuts are plain and their number is a multiple of 4.

Solution.

$$\frac{1}{1-x^4} = \frac{1}{(1-x)(1+x)(1+x^2)}$$

■

(e) The donuts must be chocolate, glazed, coconut, or plain and:

- there must be at least 3 chocolate donuts, and
- there must be at most 2 glazed, and
- there must be exactly 0 or 2 coconut, and
- there must be a multiple of 4 plain.

Solution.

$$\begin{aligned}\frac{x^3}{1-x}(1+x+x^2)(1+x^2)\frac{1}{1-x^4} &= \frac{x^3(1+x+x^2)(1+x^2)}{(1-x)^2(1+x)(1+x^2)} \\ &= \left(x^3 + x^4 + x^5\right)\frac{1}{(1-x)^2(1+x)}\end{aligned}$$

■

(f) Find a closed form for the number of ways to select n donuts subject to the constraints of the previous part.

Solution.

$$\frac{1}{(1-x)^2(1+x)} = \frac{1/2}{(1-x)^2} + \frac{1/4}{1-x} + \frac{1/4}{1+x}$$

so the nth coefficient in its generating function is

$$\frac{n+1}{2} + \frac{1}{4} + \frac{(-1)^n}{4} = \frac{2n+3+(-1)^n}{4}$$

The number ways to select n donuts is the sum of the $(n-3)$rd, $(n-4)$th, and $(n-5)$th of these coefficients, namely

$$\frac{2(n-3)+2(n-4)+2(n-5)+9+(-1)^{n-3}+(-1)^{n-4}+(-1)^{n-5}}{4} = \frac{6n-15+(-1)^{n-1}}{4}$$

■

Appendix

Products of Series

Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad B(x) = \sum_{n=0}^{\infty} b_n x^n, \qquad C(x) = A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Then

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0.$$