Solutions to the Final Examination

Problem 1 (15 points). Induction

Suppose $S(n)$ is a predicate on natural numbers, n, and suppose

$$\forall k \in \mathbb{N}\; S(k) \longrightarrow S(k+2). \tag{1}$$

If (1) holds, some of the assertions below must *always* (**A**) hold, some *can* (**C**) hold but not always, and some can *never* (**N**) hold. Indicate which case applies for each of the assertions by **circling** the correct letter.

(a) (3 points) **A N C** $(\forall n \leq 100\; S(n)) \wedge (\forall n > 100\; \neg S(n))$

Solution. N. In this case, S is true for n up to 100 and false from 101 on. So $S(99)$ is true, but $S(101)$ is false. That means that $S(k) \not\longrightarrow S(k+2)$ for $k = 99$. This case is impossible. ■

(b) (3 points) **A N C** $S(1) \longrightarrow \forall n\; S(2n+1)$

Solution. A. This assertion says that if $S(1)$ holds, then $S(n)$ holds for all odd n. This case is always true. ■

(c) (3 points) **A N C** $[\exists n\; S(2n)] \longrightarrow \forall n\; S(2n+2)$

Solution. C. If $S(n)$ is always true, this assertion holds. So

this case is possible. If $S(n)$ is true only for even n greater than 4, (1) holds, but this assertion is false. So this case does not always hold. ■

(d) (3 points) **A N C** $\exists n\, \exists m > n\, [S(2n) \wedge \neg S(2m)]$

Solution. N. This assertion says that S holds for some even number ,$2n$, but not for some other larger even number, $2m$. However, if $S(2n)$ holds, we can apply (1) $n - m$ times to conclude $S(2m)$ also holds. This case is impossible. ■

(e) (3 points) **A N C** $[\exists n\, S(n)] \longrightarrow \forall n\, \exists m > n\, S(m)$

Solution. A. This assertion says that if S holds for some n, then for every number, there is a larger number, m, for which S also holds. Since (1) implies that if there is one n for which $S(n)$ holds, there are an infinite, increasing chain of k's for which $S(k)$ holds, this case is always true. ■

Problem 2 (20 points). State Machines

We will describe a process that operates on sequences of numbers. The process will start with a sequence that is some *permutation* of the length $6n$ sequence

$$(1, 2, \ldots, n, 1, 2, \ldots, 2n, 1, 2, \ldots, 3n).$$

(a) (5 points) Write a simple formula for the number of possible starting sequences.

Solution. Using the Bookkeeper rule: There are $6n$ occurrences of digits, where each digit from 1 to n appears 3 times, and each digit from $n+1$ to $2n$ appears twice, and each digit from $2n+1$ to $3n$ appears once, so the number of possible starting sequences is

$$\frac{(6n)!}{(3!)^n(2!)^n} = \frac{(6n)!}{12^n}.$$

■

If $(s_1, \ldots, s_k)$ is a sequence of numbers, then the i and jth elements of the sequence are *out of order* if the number on the left is strictly larger the number on the right, that is, if $i < j$ and $s_i > s_j$. Otherwise, the ith and jth elements are *in order*. Define $p(S)$::= the number of "out-of-order" pairs of elements in a sequence, S.

From the starting sequence, we carry out the following process:

(*) Pick two consecutive elements in the current sequence, say the ith and $(i+1)$st.

I. If the elements are not in order, then **switch them** in the sequence and repeat step (*).

II. If the elements are in order, **remove both**, resulting in a sequence that is shorter by two. Then pick another element and remove it as well. If the length of the resulting sequence is less than three, the process is over. Otherwise, **reverse the sequence** and repeat step (*).

This process can be modelled as a state machine where the states are the sequences that appear at step (*).

(b) (5 points) Describe a simple state invariant predicate that immediately implies that if this process halts, then the final state is the sequence of length zero. (Just define the invariant; you need not prove it has the requisite properties.)

Solution. The predicate $3 \mid \text{length}(S)$ is an invariant of states, S, and is true of the start state, so is also true of the final state, so the length of the final state cannot be 1 or 2. ■

(c) (10 points) Prove that this process always terminates by defining a nonnegative integer valued derived variable that is strictly decreasing. (Just define the variable, you need not prove it has these properties.)

Solution. $6n^2 \cdot \text{length}(S) + p(S)$ as in the solution to Quiz 2, Problem 4. ■

Problem 3 (15 points). Equivalence Relations and Random Variables

A random variable, X is said to *match* a random variable, Y, iff X and Y are on the same sample space and $\Pr\{X \neq Y\} = 0$. Prove that "matches" is an equivalence relation. *Hint:* Note that $\Pr\{X \neq Z\} = \Pr\{[X \neq Z] \cap [X \neq Y]\} + \Pr\{[X \neq Z] \cap [X = Y]\}$.

Solution. Since $\Pr\{X \neq X\} = 0$, "matches" is reflexive. Also, since $\Pr\{X \neq Y\} = \Pr\{Y \neq X\}$, "matches" is symmetric.

To prove transitivity suppose X matches Y and Y matches Z.

Now we use the hint. (The hint itself follows from the fact that the event $[X \neq z]$ is the disjoint union of the events $[X \neq Z] \cap [X \neq Y]$ and $[X \neq Z] \cap [X = Y]$.) Also,

$$[X \neq Z] \cap [X \neq Y] \subseteq [X \neq Y],$$

and

$$[X \neq Z] \cap [X = Y] \subseteq [Y \neq Z],$$

so

$$\Pr\{[X \neq Z]\} \leq \Pr\{[X \neq Y]\} + \Pr\{[Y \neq Z]\}. \tag{2}$$

So if X matches Y and Y matches Z, then the righthand side of (2) is 0, and so X matches Z, which proves transivity.

An alternative proof of transitivity (not using the hint), uses the fact that if $X(s) \neq Z(s)$ at some sample point, s, then either $X(s) \neq Y(s)$ or $Y(s) \neq Z(s)$. But if $X(s) \neq Y(s)$, then $\Pr\{s\} = 0$, since X matches Y. Similarly, if $Y(s) \neq Z(s)$, then $\Pr\{s\} = 0$, since Y matches Z. So in any case, $\Pr\{s\} = 0$. So $\Pr\{X \neq Z\} = \sum_{s \in [X \neq Z]} \Pr\{s\} = \sum_{s \in [X \neq Z]} 0 = 0$, that is, X matches Z. ■

Problem 4 (15 points). Planarity.

(a) (8 points) Exhibit two planar drawings of the same 5-vertex graph in which a face in one drawing has more edges than any face in the other drawing.

Solution. One drawing is a triangle with external triangles sitting on two of its edges; the four faces are of sizes 3,3,3,5. The other drawing is gotten flipping one of the external triangles to be internal; the four faces are now of sizes 3,3,4,4. ■

(b) (7 points) Prove that all planar drawings of the same graph have the same number of faces.

Solution. We know that for any connected graph, $f = e + v - 2$ by Euler's formula. Since e and v are uniquely determined by the graph, f must also be determined.

For a general graph, we use the generalization of Euler's formula: $f + c = e + v - 1$, where c is the number of connected components of the graph. An analogous argument works.

No points were deducted for neglecting the general case. ■

Problem 5 (15 points). Inclusion-exclusion

A certain company wants to have security for their computer systems. So they have given everyone a name and password. A length 10 word containing each of the characters:

a, d, e, f, i, l, o, p, r, s,

is called a *cword*. A password will be a cword which does not contain any of the subwords "fails", "failed", or "drop".

Use the Inclusion-exclusion Principle to find a simple formula for the number of passwords.

Solution. There are $7!$ cwords that contain "drop", $6!$ that contain "fails", and $5!$ that contain "failed". There are $3!$ cwords containing both "drop" and "fails". No cword can contain both "fails" and "failed". The cwords containing both "drop" and "failed" come from taking the subword "failedrop" and the remaining letter "s" in any order, so there are $2!$ of them. So by Inclusion-exclusion, we have the number of cwords containing at least one of the three forbidden subwords is

$$(7! + 6! + 5!) - (3! + 0 + 2!) + 0 = 5!(48) - 8.$$

Among the $10!$ cwords, the remaining ones are passwords, so the number of passwords is

$$10! - 5!(48) + 8 = 3,623,048.$$

■

Problem 6 (15 points). Number Theory and Induction

(a) (5 points) Seashells are used for currency on a remote island. However, there are only *huge* shells worth 2^{10} dollars and *gigantic* shells worth 3^{12} dollars. Suppose islander A owes $m > 0$ dollars to islander B. Explain why the debt can be repaid through an exchange of shells provided A and B both have enough of each kind.

Solution. The greatest common divisor of 2^{10} and 3^{12} is 1, so there exist integers x and y such that:

$$x \cdot 2^{10} + y \cdot 3^{12} = 1$$

Multiplying both sides by m gives:

$$mx \cdot 2^{10} + my \cdot 3^{12} = m$$

Thus, islander A can repay the debt with mx huge shells and my gigantic shells. (A positive quantity indicates that A gives shells to B and a negative quantity indicates that B gives shells to A.) ■

(b) (10 points) Give an inductive proof that the Fibonacci numbers F_n and F_{n+1} are relatively prime for all $n \geq 0$. The Fibonacci numbers are defined as follows:

$$F_0 = 0, \qquad F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2} \quad \text{(for } n \geq 2\text{)}.$$

Solution. We use induction on n. Let $P(n)$ be the proposition that F_n and F_{n+1} are relatively prime.

Base case: $P(0)$ is true because $F_0 = 0$ and $F_1 = 1$ are relatively prime.

Inductive step: Assume that $P(n)$ is true where $n \geq 0$; that is, F_n and F_{n+1} are relatively prime. We must show that F_{n+1} and F_{n+2} are relatively prime as well. If F_{n+1} and F_{n+2} had a common divisor $d > 1$, then d would also divide the linear combination $F_{n+2} - F_{n+1} = F_n$, contradicting the assumption that F_n and F_{n+1} are relatively prime. So F_{n+1} and F_{n+2} are relatively prime.

The theorem follows by induction. ■

Problem 7 (20 points). Combinatorial Identities

(a) (10 points) Give a combinatorial proof that

$$\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1} \tag{3}$$

for all positive integers, n.

Solution. Consider all the teams of one or more people along with a designated team-leader from among a group of n people. There are n ways to pick the leader directly and 2^{n-1} ways to pick the remaining members of the team, for a total of $n2^{n-1}$ teams-with-leader.

Alternatively, we could pick a positive integer $i \leq n$, then a team of $i \geq 1$ people (there are $\binom{n}{i}$ ways to do this), and then pick a leader of the team (there are i ways to do that). So the total number of ways to pick a team-with-leader in this way is $\sum_{i=1}^{n} i\binom{n}{i}$. Since both processes count the number of distinct teams-with-leader, the two counts are equal.

Another way of doing this, as some students noted, is to recognize $i\binom{n}{i}$ as the total number of 1's in all n-bit strings with i 1's in them. Then $\sum_{i=1}^{n} i\binom{n}{i}$ is the total number of 1's in all possible n-bit binary strings. On the other hand, since half of the bits in the 2^n possible strings are 1's, we can conclude that this total is simply $n2^n/2 = n2^{n-1}$. ■

(b) (10 points) Now use the fact that the expected number of heads in n tosses of a fair coin is $n/2$ to give a different proof of equation (3).

Solution.

$$\begin{aligned}
\frac{n}{2} &= \mathrm{E}\left[\text{\# Heads in } n \text{ fair flips}\right] \\
&::= \sum_{i=0}^{n} i \Pr\left\{i \text{ Head in } n \text{ fair flips}\right\} \\
&= \sum_{i=1}^{n} i \binom{n}{i} \left(\frac{1}{2}\right)^{n} \\
&= \frac{\sum_{i=1}^{n} i \binom{n}{i}}{2^n}.
\end{aligned}$$

Therefore

$$n2^{n-1} = \frac{n}{2} 2^n = \sum_{i=1}^{n} i \binom{n}{i}.$$

■

Problem 8 (20 points). Generating Functions

Let a_n be the number of ways to fill a box with n doughnuts subject to the following constraints:

- The number of glazed doughnuts must be odd.
- The number of chocolate doughtnuts must be a multiple of 4.
- The number of plain doughnuts is 0 or 2.
- The number of sugar doughnuts is at most 1.

(a) (8 points) Write a generating function for each of the four doughnut types:

$$G(x) = \underline{x/(1-x^2)} \quad C(x) = \underline{1(/1-x^4)}$$

$$P(x) = \underline{1+x^2} \quad S(x) = \underline{1+x}$$

(b) (12 points) Derive a closed formula for a_n.

Solution. We have

$$\begin{aligned} A(x) &= \frac{x}{(1+x)(1-x)^2} \\ &= \frac{-1/4}{1+x} + \frac{-1/4}{1-x} + \frac{1/2}{(1-x)^2} \end{aligned}$$

by partial fractions. This gives

$$\begin{aligned} a_n &= -\frac{(-1)^n}{4} - \frac{1^n}{4} + \frac{\binom{n+1}{1}}{2} \\ &= \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n}{2} + \frac{1}{2} & \text{if } n \text{ is odd.} \end{cases} \\ &= \left\lceil \frac{n}{2} \right\rceil . \end{aligned}$$

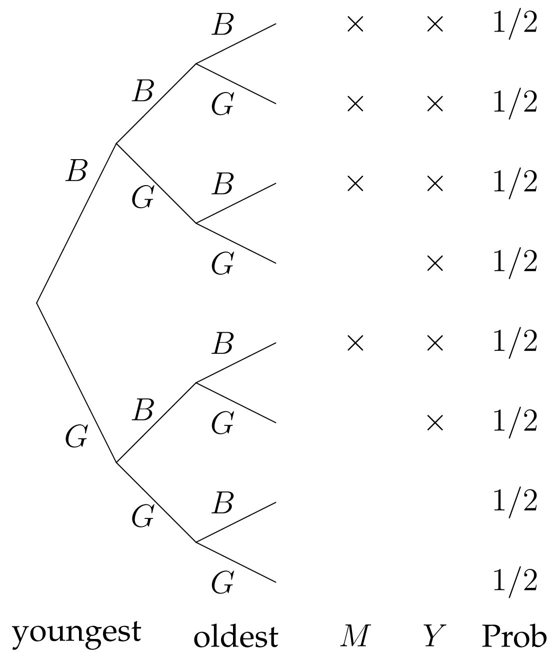
■

Problem 9 (15 points). Conditional Probability

There are 3 children of different ages. What is the probability that at least two are boys, given that at least one of the two youngest children is a boy?

Assume that each child is equally likely to be a boy or a girl and that their genders are mutually independent. A correct answer alone is sufficient. However, to be eligible for partial credit, you must include a clearly-labeled tree diagram.

Solution. Let M be the event that there are at least two boys, and let Y be the event that at least one of the two youngest children is a boy. In the tree diagram below, all edge probabilities are $1/2$.



$$\begin{aligned}
\Pr\{M \mid Y\} &= \frac{\Pr\{M \cap Y\}}{\Pr\{Y\}} \\
&= \frac{1/2}{3/4} \\
&= 2/3
\end{aligned}$$

■

Problem 10 (15 points). Probability and Expectation

A box initially contains n balls, all colored black. A ball is drawn from the box at random.

- If the drawn ball is black, then a biased coin with probability, $p > 0$, of coming up heads is flipped. If the coin comes up heads, a white ball is put into the box; otherwise the black ball is returned to the box.
- If the drawn ball is white, then it is returned to the box.

This process is repeated until the box contains n white balls.

Let D be the number of balls drawn until the process ends with the box full of white balls. Prove that $\mathrm{E}[D] = nH_n/p$, where H_n is the nth Harmonic number.

Hint: Let D_i be the number of draws after the ith white ball until the draw when the $(i+1)$st white ball is put into the box.

Solution. Suppose that the box contains k white balls. The probability that a black ball is drawn and replaced by a white ball is $p \cdot (n-k)/n$. Taking the placement of a white ball into the box as a "failure," we know the mean time to failure is $1/(p(n-k)/n) = n/p(n-k)$. So $\mathrm{E}[D_i]$, expected number of draws to introduce one more white ball is $n/p(n-k)$. By linearity of expectation, the expected number of draws required to fill the box with white balls is:

$$\begin{aligned}
\mathrm{E}[D] &= \sum_{k=0}^{n-1} D_i \\
&= \sum_{k=0}^{n-1} \frac{n}{p(n-k)} \\
&= \frac{n}{p} \cdot \sum_{k=0}^{n-1} \frac{1}{n-k} \\
&= \frac{n}{p} \cdot \sum_{j=1}^{n} \frac{1}{j} && (j = n-k) \\
&= \frac{nH_n}{p}.
\end{aligned}$$

■

Problem 11 (15 points). Deviation from the Mean

I have a randomized algorithm for calculating 6.042 grades that seems to have very strange behavior. For example, if I run it more than once on the same data, it has different running times. However, one thing I know for sure is that its *expected* running time is 10 seconds.

(a) (5 points) What does Markov's bound tell us about the probablity that my algorithm takes longer than 1 minute ($= 60$ seconds)?

Solution. the probability is $< 1/6$ ■

(b) (5 points) Suppose I decide to run the algorithm for 1 minute and if I don't get an answer by that time, I stop what I am doing, and completely restart from scratch. Each time that I stop and restart the algorithm gives me an independent run of the algorithm. So, what is an upper bound on the probability that my algorithm takes longer than 5 minutes to get an answer?

Solution. $\dfrac{1}{6^5}$ ■

(c) (5 points) Suppose some 6.042 student tells me that they determined the *variance* of the running time of my algorithm, and it is 25. What is an upper bound on the probability that my algorithm takes longer than 1 minute?

Solution. Use Chebyshev, to get $25/(50)^2 = 1/100$ ■

Problem 12 (20 points). Estimation and Confidence

On December 20, 2005, the MIT fabrication facility produced a long run of silicon wafers. To estimate the fraction, d, of defective wafers in this run, we will take a sample of n independent random choices of wafers from the run, test them for defects, and estimate that d is approximately the same as the fraction of defective wafers in the sample.

A calculation based on the Binomial Sampling Theorem (given below) will yield a near-minimal number, n_0, and such that with a sample of size $n = n_0$, the estimated fraction will be within 0.006 of the actual fraction, d, with 97% confidence.

Theorem (Binomial Sampling). *Let $K_1, K_2, \ldots,$ be a sequence of mutually independent 0-1-valued random variables with the same expectation, p, and let*

$$S_n ::= \sum_{i=1}^{n} K_i.$$

Then, for $1/2 > \epsilon > 0$,

$$\Pr\left\{\left|\frac{S_n}{n} - p\right| \geq \epsilon\right\} \leq \frac{1+2\epsilon}{2\epsilon} \cdot \frac{2^{-n(1-H((1/2)-\epsilon))}}{\sqrt{2\pi(1/4-\epsilon^2)n}} \tag{4}$$

where

$$H(\alpha) ::= -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha).$$

(a) (10 points) Explain how to use the Binomial Sampling Theorem to find n_0. You are not expected to calculate any actual values, but be sure to indicate which values should be plugged into what formulas.

Solution. To find n_0, let $\epsilon = 0.006$, and search for the smallest n such that the righthand side of (4) is ≤ 0.03. ■

(b) (10 points) The calculations in part (a) depend on some facts about the run and how the n wafers in the sample are chosen. Write T or F next to each of the following statements to indicate whether it is True or False.

- T The probability that the ninth wafer in the *sample* is defective is d.

  Solution. The ninth wafer in the sample is equally likely to be any wafer in the run, so the probability it is defective is the same as the fraction, d, of defective wafers in the fabrication run. ■

- F The probability that the ninth wafer in the *run* is defective is d.

  Solution. The fabrication run was completed yesterday, so there's nothing probabilistic about the defectiveness of the ninth (or any other) wafer in the run: either it is or it isn't defective, though we don't know which. You could argue that this means it is defective with probability zero or one (we don't know which), but in any case, it certainly isn't d. ■

- T All wafers in the run are equally likely to be selected as the third wafer in the *sample*.

  Solution. The meaning of "random choices of wafers from the run" is precisely that at each of the n choices in the sample, in particular at the third choice, each wafer in the run is equally likely to be chosen. ■

- T The expectation of the indicator variable for the last wafer in the *sample* being defective is d.

  Solution. The expectation of the indicator variable is the same as the probability that it is 1, namely, it is the probability that the nth wafer chosen is defective, which we observed in the first part of this question is d. ■

- F Given that the first wafer in the *sample* is defective, the probability that the second wafer will also be defective is less than than d.

  Solution. The meaning of "*independent* random choices of wafers from the run" is precisely that at each of the n choices in the sample, in particular at the second choice, each wafer in the run is equally likely to be chosen, independent of what the first or any other choice happened to be. ■

- F Given that the last wafer in the *run* is defective, the probability that the next-to-last wafer in the run will also be defective is less than than d.

  Solution. As noted above, it's zero or one. ■

- T It turns out that there are several different colors of wafer. Given that the first two wafers in the sample are the same color, the probability that the first wafer is defective may be $< d$.

  Solution. We don't know how color correlates to defectiveness. It could be for example, that most wafers in the run are white, and no white wafers are defective. Then given that two randomly chosen wafers in the sample are the same color, their most likely color is white. This makes them less likely to be defective than usual, that is, the conditional probability that they will be defective would be less than d. ■

- T The probability that all the wafers in the sample will be different is nonzero.

  Solution. We can assume the length, r, of the fabrication run is larger than the sample, in which case the probability that all the wafers in the sample are different is

  $$\frac{r}{r} \cdot \frac{r-1}{r} \cdot \frac{r-2}{r} \cdots \frac{r-(n-1)}{r} = \frac{r!}{(r-n)!r^n} > 0.$$

  ■

- T The probability that all choices of wafers in the sample will be different depends on the length of the run.

  Solution. The probability $r!/(r-n)!r^n$ depends on r. ■

- F The probability that all choices of wafers in the sample will be different is $\Theta(e^{-an})$ for some constant $a > 0$.

  Solution. The probability is zero once the sample is larger than the run, and zero is not $\Theta(e^{-an})$.

  Note that the Birthday Principle says that the probability that all choices of wafer are different is *approximately*

  $$e^{\frac{-n(n-1)}{2r}}.$$

  But this approximation only holds when n is not too close to r.[1] But even assuming this approximation held as n grew, the bound would not be of the form $\Theta(e^{-an})$. ■

[1] It turns out that this approximation is actually pretty good as long as $n = o(r^{2/3})$.