Solutions to Problem Set 8

Problem 1. Find the coefficients of

(a) x^{10} in $(x + (1/x))^{100}$

Solution. $x^{55}(1/x)^{45} = x^{10}$ so the coefficient is

$$\binom{100}{55}$$

■

(b) x^k in $(x^2 - (1/x))^n$.

Solution. x^k must equal $(x^2)^j(1/x)^(n - j)$ for some j where $0 \leq j \leq n$, in which case $j = (n + k)/3$. In such a case the coefficient is

$$\binom{n}{j}(-1)^{n-j} = \binom{n}{(n+k)/3}(-1)^{(2n-k)/3}.$$

■

Problem 2. Suppose a generalized World Series between the Sox and the Cardinals involved $2n + 1$ games. As usual, the generalized Series will stop as soon as one team has won more than half the possible games.

(a) Suppose that when the Sox finally win the GSeries, the Cards have managed to win *exactly* r games (so $r \leq n$). How many possible win-loss patterns are possible for the Sox to win the GSeries in this way? Express your answer as a binomial coefficient.

Solution.

$$\binom{n+r}{r} \tag{1}$$

Stars and bars, or better "S"'s and "C"'s: we can represent a win-loss pattern as a sequence of r C's and $n+1$ S's, where an S in the ith position indicates that the Sox won the ith game. However, the sequence must end with an S, so the number of such sequences is the same as the number of sequences of r C's and n S's, namely (1). ■

(b) How many possible win-loss patterns are possible for the Sox to win the GSeries when the Cards win *at most* r games? Express your answer as a binomial coefficient.

Solution.

$$\binom{n+r+1}{r} \tag{2}$$

We can represent a win-loss pattern as a sequence of r C's and $n+1$ S's, as in part (a). The number of C's which occur before the $n+1$st (last) S is the number of games the Cards won when the GSeries ends. ■

(c) Give a combinatorial proof that

$$\sum_{i=0}^{r} \binom{n+i}{i} = \binom{n+r+1}{r}. \tag{3}$$

Solution. The righthand side of (3) is the number of patterns where the Cards win at most r games. But they can at most r by winning exactly i games, where $0 \leq i \leq r$. So by part (a), the number of win-loss patterns is given by the expression of the lefthand side of (3). ■

(d) Verify equation (3) by induction using algebra.

Solution. By induction on r, taking (3) as $P(r)$.

Proof. **Base case ($r = 0$):**

$$\binom{n}{0} = 1 = \binom{n+1}{0}.$$

Inductive step:

$$\begin{aligned}
\sum_{i=0}^{r+1}\binom{n+i}{i} &= \binom{n+r+1}{r+1} + \sum_{i=0}^{r}\binom{n+i}{i} \\
&= \binom{n+r+1}{r+1} + \binom{n+r+1}{r} && \text{(by Ind. Hyp.)} \\
&= \binom{n+(r+1)+1}{r+1} && \text{(Pascal's identity),}
\end{aligned}$$

Which proves $P(r+1)$. □

■

Problem 3. **(a)** [1] Let a_n be the number of length n ternary strings (strings of the digits 0, 1, and 2) that contain two consecutive digits that are the same. For example, $a_2 = 3$ since the only ternary strings of length 2 with matching consecutive digits are `11`, `22`, and `33`. Also, $a_0 = a_1 = 0$, since in order to have consecutive matching digits, a string must be of length at least two.

Find a recurrence formula for a_n.

Solution. Call a ternary string with at least two consecutive matching digits a *good* string. Let a_n be the number of good strings of length n. Call the other $3^n - a_n$ strings of length n the *bad* strings.

Now for $n \geq 2$, a good string of length n consists of (1) a good string of length $n-1$ followed by any digit, or else (2) a bad string of length $n-1$ followed by a digit that matches the last symbol of the bad string. (Note that there is such a last symbol because $n-1 \geq 1$.) There are $3a_{n-1}$ strings of type (1) and $3^{n-1} - a_{n-1}$ strings of type (2), and these two types of strings are disjoint. So

$$a_n = 3a_{n-1} + 3^{n-1} - a_{n-1} = 2a_{n-1} + 3^{n-1}.$$

Also, a good string must have at least two digits, so $a_0 = a_1 = 0$. ■

(b) Show that

$$\frac{-x}{1-2x} + \frac{x}{(1-3x)(1-2x)}$$

is a closed form for the generating function of the sequence $a_0, a_1, \ldots$

[1] From Rosen, 5th ed., §6.1, Exercise 34.

Solution. For $n \geq 1$, the coefficient of x^n in the series expansion of $2xA(x)$ is $2a_{n-1}$, and the coefficient of x^n in

$$x + 3x^2 + 3^2x^3 + \cdots + 3^{n-1}x^n + \cdots = x(1 + 3x + (3x)^2 + \cdots + (3x)^{n-1} + \cdots) = \frac{x}{1-3x}$$

is obviously 3^{n-1}. So in the series for $A(x) - 2xA(x) - x/(1-3x)$, all the coefficients of x^n for $n \geq 2$ are zero. This leaves

$$A(x) - 2xA(x) - \frac{x}{1-3x} = (a_0 + a_1x) - 2a_0x - x = -x.$$

So

$$A(x) = \frac{-x}{1-2x} + \frac{x}{(1-3x)(1-2x)} = \frac{3x^2}{(1-3x)(1-2x)}. \tag{4}$$

■

(c) Find real numbers r and s such that

$$\frac{1}{(1-2x)(1-3x)} = \frac{r}{1-2x} + \frac{s}{1-3x}.$$

Solution. Expressing the righthand side of this equation as

$$\frac{r(1-3x) + s(1-2x)}{(1-2x)(1-3x)},$$

we need r, s so the numerators of the left and righthand expressions are equal, namely, so that

$$1 = r(1-3x) + s(1-2x).$$

So letting $x = 1/2$, we conclude that $1 = r(-1/2)$ so $r = -2$, and letting $x = 1/3$, we conclude that $1 = s(1/3)$, so $s = 3$. ■

(d) Use the previous results to write a closed form for the nth term in the sequence.

Solution. From equation (4), the generating function is

$$3x^2\left(\frac{1}{(1-2x)(1-3x)}\right) = 3x^2\left(\frac{-2}{1-2x} + \frac{3}{1-3x}\right).$$

So for $n \geq 2$, the coefficient of x^n in the generating function is $3 \cdot (-2)$ times the coefficient of x^{n-2} in $1/(1-2x)$, plus $3 \cdot 3$ times the coefficient of x^{n-2} in $1/(1-3x)$. Namely,

$$a_n = (-3 \cdot 2)2^{n-2} + 3^2 3^{n-2} = 3(3^{n-1} - 2^{n-1}).$$

■

Problem 4. Suppose there are four kinds of doughnuts: plain, chocolate, glazed, and butterscotch. Write generating functions for the number of ways to select the flavors of n doughnuts, subject to the following different constraints.

(a) Each flavor occurs an odd number of times.

Solution. Generating function for picking chocolate doughnuts is $x/(1-x^2)$, so for all 4 doughnuts it is

$$\left(\frac{x}{1-x^2}\right)^4.$$

■

(b) Each flavor occurs a multiple of 3 times.

Solution. GF for chocolate is $1/(1-x^3)$ so

$$\left(\frac{1}{1-x^3}\right)^4$$

for all 4 kinds. ■

(c) There are no chocolate doughnuts and at most one glazed doughnut.

Solution. GF for chocolate is 1, for glazed $1+x$, for others $1/(1-x)$, so for all 4 it is

$$\frac{1+x}{(1-x)^2}$$

■

(d) There are 1, 3, or 11 chocolate doughnuts, and 2, 4, or 5 glazed.

Solution. GF for chocolate is $x+x^3+x^{11}$, for glazed $x^2+x^4+x^5$, and $1/(1-x)$ for the others, so

$$\frac{(x+x^3+x^{11})(x^2+x^4+x^5)}{(1-x)^2}.$$

for all. ■

(e) Each flavor occurs at least 10 times.

Solution. GF for chocolate is $x^{10}/(1-x)$, so

$$\frac{x^{40}}{(1-x)^4}$$

for all 4. ■