Solutions to Problem Set 4

Problem 1. For functions $f : A \to B$ and $g : B \to C$, the composition of g and f, written $g \circ f$, is the function $h : A \to C$ where

$$h(a) ::= g(f(a)).$$

(a) Prove that if f and g are bijections, then so is $g \circ f$.

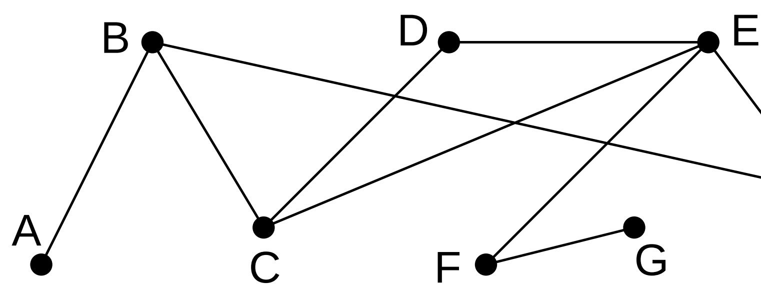
(b) Prove that if $f : A \to B$ is a bijection, then there is a bijection, $e : B \to A$ such that $e \circ f = I_A$, where $I_A : A \to A$ and $I_A(a) ::= a$ for all $a \in A$.

(c) Prove that graph isomorphism is an equivalence.

Problem 2. The proof of the Handshake Theorem in Week 5 Notes is a little more informal than is desirable in the beginning of 6.042. Rewrite the proof more carefully as an induction on the number of edges in a graph.

Problem 3. The ***distance*** between two vertices in a graph is the length of the shortest path between them. For example, the distance between two vertices in a graph of airline connections is the minimum number of flights required to travel between two cities. The ***diameter*** of a graph is the distance between the two vertices that are farthest apart.

(a) What is the diameter of the following graph? Briefly explain your answer.



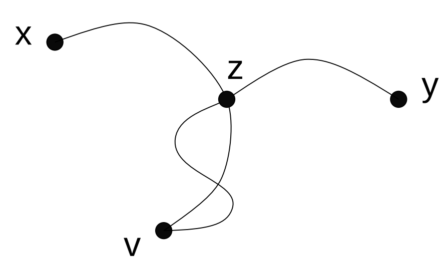
Solution. The diameter of the graph above is 5. The most-distant vertices are A and G, which are at distance 5 from one another. ■

(b) What is the chromatic number of this graph? Prove it.

Solution. It's easy to find a 3-coloring, for example, let C, D, E be colored c, d, e, then color B, H and F with color d, and G and H with e. The chromatic number cannot be less than 3, because vertices C, D, E are all connected and therefore must receive distinct colors. Consequently, the chromatic number of the graph is exactly 3. ■

(c) Suppose every vertex in a graph is within distance n of a vertex, v. Prove that the diameter of the graph is at most $2n$.

Solution. Let x and y be arbitrary vertices in the graph. Then there exists a path of length at most d from x to v, and there exists a path of length at most d from v to y.



Let z be the vertex that lies on both the x-to-v and v-to-y paths and is closest to x. (We know that such a vertex exists, since z could be v, at least.) Joining the x-to-z segment to the z-to-y segment gives a path from x to y of length at most $2n$. Therefore, every vertex is within distance $2n$ of every other. ■

Problem 4. If a graph is connected, then every vertex must be adjacent to some other vertex. Is the converse of this statement true? If every vertex is adjacent to some other vertex, then is the graph connected? The answer is no.

(a) Give a minimal example of a graph in which every vertex is adjacent to at least one other vertex, but the graph is not connected.

(b) So something is wrong with the following proof. Exactly where is the first mistake in the proof?

False Theorem 4.1. *If every vertex in a graph is adjacent to another vertex, then the graph is connected.*

Nothing helps a false proof like a good picture; see Figure 1.

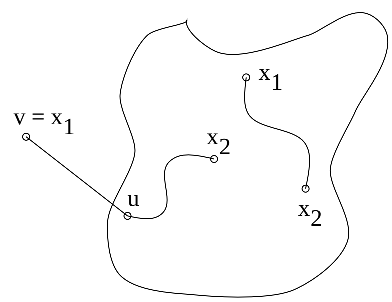


Figure 1: *This picture accompanies the false proof. Two situations are depicted. In one, vertices x_1 and x_2 both are among the vertices of G, and so there is a connecting path by induction. In the second, $v = x_1$ and x_2 is a vertex of G. In this case there is a connecting path because there is an edge from v to u and a path in G from u to x_2 by induction.*

Proof. The proof is by induction. Let $P(n)$ be the predicate that if every vertex in an n-vertex graph is adjacent to another vertex, then the graph is connected. In the base case, $P(1)$ is trivially true because there is only one vertex.

In the inductive step, we assume $P(n)$ to prove $P(n+1)$. Start with an $n+1$-vertex graph, G', in which every vertex is adjacent to another vertex. Now take some vertex v away from the graph and let the G be the remaining graph. By assumption v is adjacent in G' to one of the n vertices of G; call that one u.

Now we must show that for every pair of distinct vertices x_1 and x_2 in G', there is a path between them. If both x_1 and x_2 are vertices of G, then since G has n vertices, we may assume by induction it is connected. So there is a path between x_1 and x_2. Otherwise, one of the vertices is v (say x_1) and the other, x_2 is in G. But x_2 is connected to u by induction, so there is a path from x_1 to u to x_2 as shown in the figure. □

Solution. The error is in the statement *"since G has n vertices, we may assume by induction it is connected."* The induction hypothesis does not say that every n-vertex graph is connected, but only, *"if every vertex in an n-vertex graph is adjacent to another vertex,* then the graph is connected". For example, if G' is the graph with vertices 1, 2, 3, 4 and edges $\{1,2\}$ and $\{3,4\}$, then removing vertex 1 to form G leaves vertex 2 without an adjacent vertex in G, and we can't conclude by induction that G is connected (which of course it isn't). ■

Problem 5. (a) Show that every planar graph has a node of degree at most 5. *Hint:* Use the $3v - 6$ inequality.

Solution. Consider some planar graph G. Take any connected component H of G. If H has at most 2 vertices, then every vertex in H has degree at most 1. Otherwise, for the sake of contradiction suppose that every vertex in H has degree at least 6. Then

$$2e = \sum_{n \in V} \deg(n) \geq \sum_{n \in V} 6 = 6v > 6v - 12,$$

contradicting the fact that $e \leq 3v - 6$ in any connected planar graph with more than 2 vertices. ■

(b) Conclude that *any* planar graph can be colored with six colors.

Solution. The proof is almost identical to the proof in Week 5 Notes that a graph with maximum degree d is $d+1$ colorable.

The proof is by strong induction on the number, v, of vertices. The induction hypothesis is

$P(v)$: A planar graph with v vertices can be colored with at most 6 colors.

Base cases ($v \leq 6$): $P(v)$ is true because each vertex can be assigned a different color.

Inductive step: Assume $P(v)$ is true for some $v \geq 6$ and show that $P(v+1)$ is true.

Let G be a planar graph with $v+1$ vertices. By part (a), there is a node, n, of degree ≤ 5. Let G' be the subgraph of G obtained by removing node n and all edges incident to it.

Now G' has v-vertices and is also planar, so by induction hypothesis, G' can be colored with 6 colors. Now re-attach n. Since n is adjacent to at most 5 vertices, there is a 6th color for n different from colors of its adjacent vertices. This yields a 6-coloring of G.

We have shown that $P(v) \to P(v+1)$, so the proof is complete. ■