

## Problem Set 4

*Due:* October 12

**Reading:** Week 5 Notes and Undirected Graphs section of Week 4 Notes

**Problem 1.** For functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , the composition of  $g$  and  $f$ , written  $g \circ f$ , is the function  $h : A \rightarrow C$  where

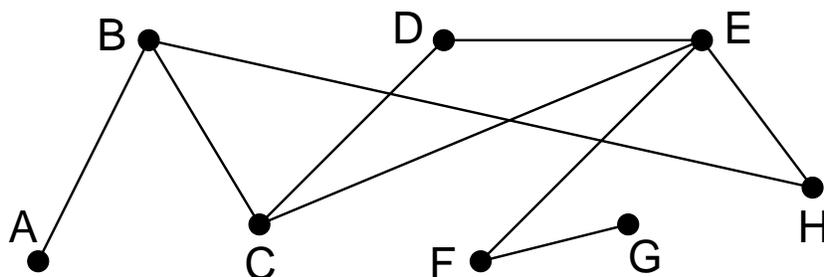
$$h(a) ::= g(f(a)).$$

- (a) Prove that if  $f$  and  $g$  are bijections, then so is  $g \circ f$ .
- (b) Prove that if  $f : A \rightarrow B$  is a bijection, then there is a bijection,  $e : B \rightarrow A$  such that  $e \circ f = I_A$ , where  $I_A : A \rightarrow A$  and  $I_A(a) ::= a$  for all  $a \in A$ .
- (c) Prove that graph isomorphism is an equivalence.

**Problem 2.** The proof of the Handshake Theorem in Week 5 Notes is a little more informal than is desirable in the beginning of 6.042. Rewrite the proof more carefully as an induction on the number of edges in a graph.

**Problem 3.** The *distance* between two vertices in a graph is the length of the shortest path between them. For example, the distance between two vertices in a graph of airline connections is the minimum number of flights required to travel between two cities. The *diameter* of a graph is the distance between the two vertices that are farthest apart.

- (a) What is the diameter of the following graph? Briefly explain your answer.



(b) What is the chromatic number of this graph? Prove it.

(c) Suppose every vertex in a graph is within distance  $n$  of a vertex,  $v$ . Prove that the diameter of the graph is at most  $2n$ .

**Problem 4.** If a graph is connected, then every vertex must be adjacent to some other vertex. Is the converse of this statement true? If every vertex is adjacent to some other vertex, then is the graph connected? The answer is no.

(a) Give a minimal example of a graph in which every vertex is adjacent to at least one other vertex, but the graph is not connected.

(b) So something is wrong with the following proof. Exactly where is the first mistake in the proof?

**False Theorem 4.1.** *If every vertex in a graph is adjacent to another vertex, then the graph is connected.*

Nothing helps a false proof like a good picture; see Figure 1.

*Proof.* The proof is by induction. Let  $P(n)$  be the predicate that if every vertex in an  $n$ -vertex graph is adjacent to another vertex, then the graph is connected. In the base case,  $P(1)$  is trivially true because there is only one vertex.

In the inductive step, we assume  $P(n)$  to prove  $P(n + 1)$ . Start with an  $n + 1$ -vertex graph,  $G'$ , in which every vertex is adjacent to another vertex. Now take some vertex  $v$  away from the graph and let the  $G$  be the remaining graph. By assumption  $v$  is adjacent in  $G'$  to one of the  $n$  vertices of  $G$ ; call that one  $u$ .

Now we must show that for every pair of distinct vertices  $x_1$  and  $x_2$  in  $G'$ , there is a path between them. If both  $x_1$  and  $x_2$  are vertices of  $G$ , then since  $G$  has  $n$  vertices, we may assume by induction it is connected. So there is a path between  $x_1$  and  $x_2$ . Otherwise, one of the vertices is  $v$  (say  $x_1$ ) and the other,  $x_2$  is in  $G$ . But  $x_2$  is connected to  $u$  by induction, so there is a path from  $x_1$  to  $u$  to  $x_2$  as shown in the figure. □

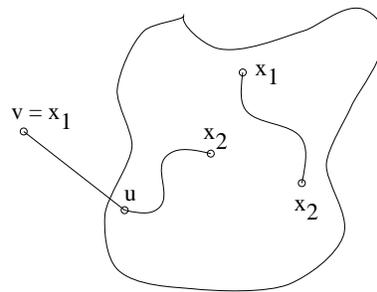


Figure 1: This picture accompanies the false proof. Two situations are depicted. In one, vertices  $x_1$  and  $x_2$  both are among the vertices of  $G$ , and so there is a connecting path by induction. In the second,  $v = x_1$  and  $x_2$  is a vertex of  $G$ . In this case there is a connecting path because there is an edge from  $v$  to  $u$  and a path in  $G$  from  $u$  to  $x_2$  by induction.

**Problem 5. (a)** Show that every planar graph has a node of degree at most 5. *Hint:* Use the  $3v - 6$  inequality.

**(b)** Conclude that *any* planar graph can be colored with six colors.

## Student's Solutions to Problem Set 4

**Your name:**

**Due date:** October 12

**Submission date:**

**Circle your TA:** David Jelani Sayan Hanson

**Collaboration statement:** Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:  
got help from:<sup>1</sup>  
and referred to:<sup>2</sup>

---

**DO NOT WRITE BELOW THIS LINE**

---

Problem	Score
1	
2	
3	
4	
5	
Total	

---

Copyright © 2005, Prof. Albert R. Meyer and Prof. Ronitt Rubinfeld.

<sup>1</sup>People other than course staff.

<sup>2</sup>Give citations to texts and material other than the Fall '02 course materials.