

## Solutions to Problem Set 2

**Problem 1.** Use induction to prove that the following inequality holds for all integers  $n \geq 1$ .

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} \geq \frac{1}{2n+2}$$

**Solution.** We use induction. Let  $P(n)$  be the proposition that:

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} \geq \frac{1}{2n+2}$$

First, we prove  $P(0)$ . In this case, both side of the inequality are equal to  $1/2$ , and so the inequality holds. Next, for each  $n \geq 0$ , we must show that  $P(n)$  implies  $P(n+1)$ . Assume that  $P(n)$  is true. Then we can reason as follows:

$$\begin{aligned} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)(2n+3)}{2 \cdot 4 \cdot 6 \cdots (2n+2)(2n+4)} &\geq \frac{1}{2n+2} \cdot \frac{2n+3}{2n+4} \\ &> \frac{1}{2n+4} \end{aligned}$$

The first step uses the induction hypothesis,  $P(n)$ . The second step uses the fact that  $(2n+3)/(2n+2) > 1$  for all  $n \geq 1$ . Therefore, by induction, the proposition  $P(n)$  is true for all  $n \geq 1$ , and the claim is proved. ■

**Problem 2.** This term in 6.042, we're constantly trying to divide a class of  $n$  students into groups of either 4 or 5 students.

(a) Let's try to use strong induction prove that a class with  $n \geq 8$  students can be divided into groups of 4 or 5.

*Proof.* The proof is by strong induction. Let  $P(n)$  be the proposition that a recitation with  $n$  students can be divided into teams of 4 or 5.

First, we prove that  $P(n)$  is true for  $n = 8, 9,$  or  $10$  by showing how to break classes of these sizes into groups of 4 or 5 students:

$$\begin{aligned} 8 &= 4 + 4 \\ 9 &= 4 + 5 \\ 10 &= 5 + 5 \end{aligned}$$

Next, we must show that  $P(8), \dots, P(n)$  imply  $P(n + 1)$  for all  $n \geq 10$ . Thus, we assume that  $P(8), \dots, P(n)$  are all true, and show how to divide up a class of  $n + 1$  students into groups of 4 or 5. We first form one group of 4 students. Then we can divide the remaining  $n - 3$  students into groups of 4 or 5 by the assumption  $P(n - 3)$ . This proves  $P(n + 1)$ , and so the claim holds by induction.  $\square$

This proof contains a critical logical error. (In fact, the claim is false!) Identify the first sentence in the proof that does not follow and explain what went wrong.

**Solution.** The first error is in the sentence:

Then we can divide the remaining  $n - 3$  students into groups of 4 or 5 by the assumption  $P(n - 3)$ .

If  $n = 10$ , then  $P(n - 3) = P(7)$ , which is not among our assumptions  $P(8), \dots, P(n)$ . In this case,  $P(n + 1) = P(11)$  is actually false.  $\blacksquare$

(b) Provide a correct strong induction proof that a class with  $n \geq 12$  students can be divided into groups of 4 or 5.

**Solution.** The proof is by strong induction. Let  $P(n)$  be the proposition that a recitation with  $n$  students can be divided into teams of 4 or 5.

First, we prove that  $P(n)$  is true for  $n = 12, 13, 14,$  and  $15$  by showing how to break classes of these sizes into groups of 4 or 5 students:

$$\begin{aligned} 12 &= 4 + 4 + 4 \\ 13 &= 4 + 4 + 5 \\ 14 &= 4 + 5 + 5 \\ 15 &= 5 + 5 + 5 \end{aligned}$$

Next, we must show that  $P(12), \dots, P(n)$  imply  $P(n+1)$  for all  $n \geq 15$ . Thus, we assume that  $P(12), \dots, P(n)$  are all true and show how to divide up a class of  $n+1$  students. We first form one group of 4 students. Then we can divide the remaining  $n-3$  students into groups of 4 or 5 by the assumption  $P(n-3)$ . (Note that  $n \geq 15$  and so  $n-3 \geq 12$ ; thus,  $P(n-3)$  is among our assumptions  $P(12), \dots, P(n)$ .) This proves  $P(n+1)$ , and so the claim holds by induction. ■

**Problem 3.** The game of *Mini-nim* is defined as follows: Some positive number of sticks are placed on the ground. Two players take turns removing one, two, or three sticks. The player to remove the last one loses.

Use strong induction to show that:

*The second player has a winning strategy if the number of sticks, equals  $4k+1$  for some  $k \in \mathbb{N}$ ; otherwise, the first player has a winning strategy.*

**Solution.** The induction hypothesis,  $P(n)$ , is:

If  $n = 4k+1$ , for some  $k \in \mathbb{N}$ , then the second player has a winning strategy; otherwise, the first player has a winning strategy.

We proceed by strong induction, starting from 1.

**Base case:**  $n = 1$ . The first player has no choice but to remove 1 stick and lose, which is what the theorem says for this case.

**Strong inductive step:** Suppose the theorem is true for numbers 1 through  $n$  and show that it is true for  $n+1$ . For the inductive step, there are four cases:

- $n+1 = 4k+1$ : show that the first player loses. We've already handled the base case (1) so we can assume  $n+1 \geq 5$ . Consider what the first player might do to win: he can choose to remove 1, 2 or 3 sticks. If he removes one stick, the remaining number of sticks is  $n = 4k$ . By strong induction, the player who plays at this point has a winning strategy. So the player who played first will lose.

Similarly, if the first player removes two sticks, the remaining number is  $4(k-1)+3$ . Again, he loses, by the same reasoning. Similarly, by removing 3 sticks, he loses. So, however the first player moves, he loses.

- $n + 1 = 4k$ : show that the first player can win.  
Have the first player remove 3 sticks: the second player then sees  $4(k - 1) + 1$  sticks, and loses, by the strong inductive hypothesis.
- $n + 1 = 4k + 2$ : show that the first player can win.  
Have the first player remove 1 stick: the second player then sees  $4k + 1$  sticks, and loses as in the previous case.
- $n + 1 = 4k + 3$ : show that the first player can win.  
Have the first player remove 2 sticks: again, the second player sees  $4k + 1$  sticks and loses.

So in any case,  $P(n + 1)$  is true, and by strong induction, we conclude that  $P(n)$  holds for all  $n \geq 1$ . ■

**Problem 4.** Consider the following equivalent way of viewing the subset take-away game from the in-class problem on Friday, Week 2: for a fixed, finite set,  $A$ , let  $\mathcal{S}$  initially be all the proper subsets of  $A$ . Players alternately choose a set  $B \in \mathcal{S}$  and remove  $B$  and all sets that contain  $B$  from  $\mathcal{S}$ ; they then continue playing on the updated  $\mathcal{S}$ . The player that chooses the last set in  $\mathcal{S}$  wins.

(a) Use the well-ordering property to show that, in any game, one of the players must have a winning strategy. *Hint:* Consider games whose initial set,  $\mathcal{S}$ , is an arbitrary collection of subsets of,  $A$ , not necessarily all the proper subsets of  $A$ . Reach a contradiction by considering a minimum size game with no winning strategy for either player. What is a useful measure of size of a game?

**Solution.** Let  $S$  be the smallest collection such that neither player has a winning strategy. Consider the state after each possible first move by player 1. No such state can correspond to a winning position for player 1 since that would mean a winning strategy for player 1 on  $S$ . Nor can every such state be a winning position for player 2 since that would mean player 2 has a winning strategy on  $S$ . Hence, at least one such state must have no winning strategies for both players. But this state is a smaller collection than  $S$ , contradiction. Thus, by the W.O.P, some player must have a winning strategy for any collection  $S$ , and for the proper subsets of  $A$  in particular. ■

(b) If the whole set  $A$  is a possible move in a game, explain why the 1st player must have a winning strategy.

**Solution.** We reason by cases to show that player 1 has a winning strategy. Suppose game  $G$  includes  $A$  as a possible move. Let  $G'$  be the same game as  $G$  except that  $A$  is removed from the set of possible moves.

**Case 1:** Player 1 has a winning strategy in the game  $G'$ . Then the first move of Player 1's winning strategy will also be a legal move in game  $G$ . Moreover, after this move in game  $G$ , the set  $A$  will no longer be a possible move, so the move will lead to the same winning situation for Player 1 as in game  $G'$ . So Player 1 has a winning strategy in this case.

**Case 2:** Player 2 has a winning strategy in game  $G'$ . Then player 1 should choose  $A$  as his first move in game  $G$ . This puts him in the position of the Player 2 in game  $G'$ , and so Player 1 will have a winning strategy in this case as well.

So in any case, we conclude that Player 1 has a winning strategy—even though we don't have a clue what it is. ■