Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 21 Solutions
November 23, 2010

1. (a) To use the Markov inequality, let $X = \sum_{i=1}^{10} X_i$. Then,

$$\mathbf{E}[X] = 10\mathbf{E}[X_i] = 5,$$

and the Markov inequality yields

$$\mathbf{P}(X \geq 7) \leq \frac{5}{7} = 0.7142.$$

(b) Using the Chebyshev inequality, we find that

$$\begin{aligned} 2\mathbf{P}(X - 5 \geq 2) &= \mathbf{P}(|X - 5| \geq 2) \\ &\leq \frac{\text{var}(X)}{4} = \frac{10/12}{4} \\ \mathbf{P}(X - 5 \geq 2) &\leq \frac{5}{48} = 0.1042. \end{aligned}$$

(c) Finally, using the Central Limit Theorem, we find that

$$\begin{aligned} \mathbf{P}\left(\sum_{i=1}^{10} X_i \geq 7\right) &= 1 - \mathbf{P}\left(\sum_{i=1}^{10} X_i \leq 7\right) \\ &= 1 - \mathbf{P}\left(\frac{\sum_{i=1}^{10} X_i - 5}{\sqrt{10/12}} \leq \frac{7-5}{\sqrt{10/12}}\right) \\ &\approx 1 - \Phi(2.19) \\ &\approx 0.0143. \end{aligned}$$

2. Check online solutions.

3. (a) If we interpret X_i as the number of arrivals in an interval of length 1 in a Poisson process of rate 1, then, $S_n = X_1 + \cdots + X_n$ can be seen as the number of arrivals in an interval of length n in the Poisson process of rate 1. Therefore, S_n is a Poisson random variable with mean and variance equal to n.

(b) We use the random variables $X_1, \ldots, X_n$ and the random variable $S_n = X_1 + \cdots + X_n$. Denoting by Z the standard normal, and applying the central limit theorem, we have for large n

$$\begin{aligned}
\mathbf{P}(S_n = n) &= \mathbf{P}(n - 1/2 < S_n < n + 1/2) \\
&= \mathbf{P}\left(\frac{-1}{2\sqrt{n}} < \frac{S_n - n}{\sqrt{n}} \leq \frac{1}{2\sqrt{n}}\right) \\
&\approx \mathbf{P}\left(\frac{-1}{2\sqrt{n}} < Z \leq \frac{1}{2\sqrt{n}}\right) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1/2\sqrt{n}}^{1/2\sqrt{n}} e^{-z^2/2} dz \\
&\approx \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n}} e^{-z^2/2} \Big|_{z=0} \\
&= \frac{1}{\sqrt{2\pi n}}
\end{aligned}$$

where the first equation follows from the fact that S_n takes integer values, the first approximation is suggested by the central limit theorem, and the second approximation uses the fundamental theorem of calculus (the value of a definite integral over a small interval is equal to the length of the interval times the integrand evaluated at some point within the interval). Since S_n is Poisson with mean n, we have

$$\mathbf{P}(S_n = n) = e^{-n} \frac{n^n}{n!},$$

and by combining the preceding relations, we see that $n! \approx n^n e^{-n} \sqrt{2\pi n} = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$.
One may show that

$$\lim_{n \to \infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1,$$

so the relative error of the approximation tends to 0 as $n \to \infty$. A more precise estimate is that

$$n! = n^n e^{-n} \sqrt{2\pi n} \cdot e^{\lambda_n},$$

where

$$\frac{1}{12n + 1} < \lambda_n < \frac{1}{12n}.$$

However, one cannot derive these relations from the central limit theorem.

Note that the form of the approximation was first discovered by de Moivre in the form $n! \approx n^{n+1/2} e^{-n} \cdot$(constant), and gave a complicated expression for the constant. De Moivre's friend Stirling subsequently showed that the constant has the simple form $\sqrt{2\pi}$.