

6.041/6.431 Probabilistic Systems Analysis

Quiz II Review
Fall 2010

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1 Probability Density Functions (PDF)

For a continuous RV X with PDF $f_X(x)$,

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$
$$P(X \in A) = \int_A f_X(x) dx$$

Properties:

- Nonnegativity:

$$f_X(x) \geq 0 \quad \forall x$$

- Normalization:

$$\int_{-\infty}^{\infty} f_X(x) dx = 1$$

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2 PDF Interpretation

Caution: $f_X(x) \neq P(X = x)$

- if X is continuous, $P(X = x) = 0 \quad \forall x!!$
- $f_X(x)$ can be ≥ 1

Interpretation: “probability per unit length” for “small” lengths around x

$$P(x \leq X \leq x + \delta) \approx f_X(x)\delta$$

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3 Mean and variance of a continuous RV

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$\begin{aligned} \text{Var}(X) &= E[(X - E[X])^2] \\ &= \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) dx \\ &= E[X^2] - (E[X])^2 \quad (\geq 0) \end{aligned}$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

$$E[aX + b] = aE[X] + b$$

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

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4 Cumulative Distribution Functions

Definition:

$$F_X(x) = P(X \leq x)$$

monotonically increasing from 0 (at $-\infty$) to 1 (at $+\infty$).

- Continuous RV (CDF is continuous in x):

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t) dt$$

$$f_X(x) = \frac{dF_X}{dx}(x)$$

- Discrete RV (CDF is piecewise constant):

$$F_X(x) = P(X \leq x) = \sum_{k \leq x} p_X(k)$$

$$p_X(k) = F_X(k) - F_X(k-1)$$

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5 Uniform Random Variable

If X is a uniform random variable over the interval $[a, b]$:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{otherwise (} x > b \text{)} \end{cases}$$

$$E[X] = \frac{b-a}{2}$$

$$\text{var}(X) = \frac{(b-a)^2}{12}$$

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6 Exponential Random Variable

X is an exponential random variable with parameter λ :

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \frac{1}{\lambda} \quad \text{var}(X) = \frac{1}{\lambda^2}$$

Memoryless Property: Given that $X > t$, $X - t$ is an exponential RV with parameter λ

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7 Normal/Gaussian Random Variables

General normal RV: $N(\mu, \sigma^2)$:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$E[X] = \mu, \quad \text{Var}(X) = \sigma^2$$

Property: If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$

then $Y \sim N(a\mu + b, a^2\sigma^2)$

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8 Normal CDF

Standard Normal RV: $N(0, 1)$

CDF of standard normal RV Y at y : $\Phi(y)$

- given in tables for $y \geq 0$

- for $y < 0$, use the result: $\Phi(y) = 1 - \Phi(-y)$

To evaluate CDF of a general standard normal, express it as a function of a standard normal:

$$X \sim N(\mu, \sigma^2) \Leftrightarrow \frac{X - \mu}{\sigma} \sim N(0, 1)$$

$$P(X \leq x) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

9 Joint PDF

Joint PDF of two continuous RV X and Y : $f_{X,Y}(x, y)$

$$P(A) = \int \int_A f_{X,Y}(x, y) dx dy$$

Marginal pdf: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$

$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy$

Joint CDF: $F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$

10 Independence

By definition,

$$X, Y \text{ independent} \Leftrightarrow f_{X,Y}(x, y) = f_X(x)f_Y(y) \quad \forall(x, y)$$

If X and Y are independent:

- $E[XY] = E[X]E[Y]$
- $g(X)$ and $h(Y)$ are independent
- $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$

11 Conditioning on an event

Let X be a continuous RV and A be an event with $P(A) > 0$,

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(X \in A)} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

$$P(X \in B | X \in A) = \int_B f_{X|A}(x) dx$$

$$E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x) dx$$

$$E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x) dx$$

If A_1, \dots, A_n are disjoint events that form a partition of the sample space,

$$f_X(x) = \sum_{i=1}^n P(A_i) f_{X|A_i}(x) \quad (\approx \text{total probability theorem})$$

$$E[X] = \sum_{i=1}^n P(A_i) E[X|A_i] \quad (\text{total expectation theorem})$$

$$E[g(X)] = \sum_{i=1}^n P(A_i) E[g(X)|A_i]$$

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12 Conditioning on a RV

X, Y continuous RV

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_Y(y) f_{X|Y}(x|y) dy \quad (\approx \text{total probthm})$$

Conditional Expectation:

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

$$E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y) dx$$

$$E[g(X,Y)|Y=y] = \int_{-\infty}^{\infty} g(x,y) f_{X|Y}(x|y) dx$$

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Total Expectation Theorem:

$$E[X] = \int_{-\infty}^{\infty} E[X|Y=y] f_Y(y) dy$$

$$E[g(X)] = \int_{-\infty}^{\infty} E[g(X)|Y=y] f_Y(y) dy$$

$$E[g(X,Y)] = \int_{-\infty}^{\infty} E[g(X,Y)|Y=y] f_Y(y) dy$$

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13 Continuous Bayes' Rule

X, Y continuous RV, N discrete RV, A an event.

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} = \frac{f_{Y|X}(y|x) f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|t) f_X(t) dt}$$

$$P(A|Y=y) = \frac{P(A) f_{Y|A}(y)}{f_Y(y)} = \frac{P(A) f_{Y|A}(y)}{f_{Y|A}(y) P(A) + f_{Y|A^c}(y) P(A^c)}$$

$$P(N=n|Y=y) = \frac{p_N(n) f_{Y|N}(y|n)}{f_Y(y)} = \frac{p_N(n) f_{Y|N}(y|n)}{\sum_i p_N(i) f_{Y|N}(y|i)}$$

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14 Derived distributions

Def: PDF of a *function* of a RV X with known PDF: $Y = g(X)$.

Method:

- Get the CDF:

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \int_{x|g(x) \leq y} f_X(x) dx$$

- Differentiate: $f_Y(y) = \frac{dF_Y}{dy}(y)$

Special case: if $Y = g(X) = aX + b$, $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

15 Convolution

$W = X + Y$, with X, Y independent.

- Discrete case:

$$p_W(w) = \sum_x p_X(x) p_Y(w - x)$$

- Continuous case:

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x) dx$$

Graphical Method:

- put the PMFs (or PDFs) on top of each other
- flip the PMF (or PDF) of Y
- shift the flipped PMF (or PDF) of Y by w
- cross-multiply and add (or evaluate the integral)

In particular, if X, Y are independent and normal, then $W = X + Y$ is normal.

16 Law of iterated expectations

$E[X|Y = y] = f(y)$ is a number.

$E[X|Y] = f(Y)$ is a random variable

(the expectation is taken with respect to X).

To compute $E[X|Y]$, first express $E[X|Y = y]$ as a function of y .

Law of iterated expectations:

$$E[X] = E[E[X|Y]]$$

(equality between two real numbers)

17 Law of Total Variance

$\text{Var}(X|Y)$ is a random variable that is a function of Y (the variance is taken with respect to X).

To compute $\text{Var}(X|Y)$, first express

$$\text{Var}(X|Y = y) = E[(X - E[X|Y = y])^2|Y = y]$$

as a function of y .

Law of conditional variances:

$$\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}(E[X|Y])$$

(equality between two real numbers)

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18 Sum of a random number of iid RVs

N discrete RV, X_i i.i.d and independent of N .

$Y = X_1 + \dots + X_N$. Then:

$$E[Y] = E[X]E[N]$$

$$\text{Var}(Y) = E[N]\text{Var}(X) + (E[X])^2\text{Var}(N)$$

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19 Covariance and Correlation

$$\begin{aligned}\text{Cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - E[X]E[Y]\end{aligned}$$

- By definition, X, Y are uncorrelated $\Leftrightarrow \text{Cov}(X, Y) = 0$.
- If X, Y independent $\Rightarrow X$ and Y are uncorrelated. (the converse is not true)
- In general, $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2 \text{Cov}(X, Y)$
- If X and Y are uncorrelated, $\text{Cov}(X, Y) = 0$ and $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$

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Correlation Coefficient: (dimensionless)

$$\rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} \in [-1, 1]$$

$\rho = 0 \Leftrightarrow X$ and Y are uncorrelated.

$|\rho| = 1 \Leftrightarrow X - E[X] = c[Y - E[Y]]$ (linearly related)

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