

6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Recitation: A Mixed Distribution Example

In this video, we'll look at an example in which we compute the expectation and cumulative density function of a mixed random variable. The problem is as follows. Al arrives at some bus stand or taxi stand at a given time-- let's say time t equals 0. He finds a taxi waiting for him with probability $2/3$ in which he takes it. Otherwise, he takes the next arriving taxi or bus. The time that the next taxi arrives between 0 and 10 minutes, and it's uniformly distributed. The next bus leaves exactly in 5 minutes. So the question is, if X is Al's waiting time, what is the CDF and expectation of X ?

So one way to view this problem that's convenient is the tree structure. So I've drawn it for you here in which the events of interest are B_1 , B_2 , and B_3 , B_1 being Al catches the waiting taxi, B_2 being Al catches the next taxi, which arrives between 0 and 5 minutes, and B_3 being Al catches the bus at the time t plus 5. Notice that these three events are disjoint. So Al catching the waiting taxi means he can't catch the bus or the next arriving taxi. And it also covers the entire set of outcomes. So, in fact, B_1 , B_2 , and B_3 are a partition.

So let's look at the relevant probabilities. Whether or not B_1 happens depends on whether or not the taxi's waiting for Al. So if the taxi is waiting for him, which happens with $2/3$ probability, B_1 happens. Otherwise, with $1/3$ probability, we see whether or not a taxi is going to arrive between 0 and 5 minutes. If it arrives, which is going to happen with what probability? Well, we know that the next taxi is going to arrive between 0 and 10 minutes uniform. It's a uniform distribution. And so half the mass is going to be between 0 and 5. And the other half is going to be between 5 and 10. And so this is going to be $1/2$ and $1/2$.

And let's look at what X looks like. If B_1 happens, Al isn't waiting at all, so x is going to be equal to 0. If B_3 happens, which is the other easy case, Al's going to be waiting for 5 minutes exactly. And if B_2 happens, well, it's going to be some value between 0 and 5.

We can actually draw the density, so let's see if we can do that here. The original next taxi was uniformly distributed between 0 and 10. But now, we're told two pieces of information. We're told that B_2 happens, which means that there's no taxi waiting, and the next taxi arrives between 0 and 5 minutes. Well, the fact that there was no taxi waiting has no bearing on that density. But the fact that the next taxi arrives between 0 and 5 does make a difference, because the density then is going to be definitely 0 in any region outside 0 and 5.

Now, the question is, how is it going to look between 0 and 5? Well, it's not going to look crazy. It's not going to look like something different. It's simply going to be a scale version of the original density between 0 and 5. You can verify this by looking at the actual formula for when you condition events on a random variable. Here, it's going to be $1/5$ in order for this to integrate out to 1. And now we can jump right into figuring out the expectation.

Now, notice that X is actually a mixed random variable? What does that mean? Well, X either takes on values according to either a discrete probability law or a continuous one. So if B_1

happens, for example, X is going to be exactly equal to 0 with probability 1, which is a discrete probability problem. On the other hand, if B_2 happens, then the value of X depends on the density, which is going to be continuous. So X is going to be a continuous random variable here.

So how do you define an expectation in this case? Well, you can do it so that it satisfies the total expectation theorem, which means that the expectation of X is the probability of B_1 times the expectation given B_1 plus the probability of B_2 times the expectation given B_2 plus the probability of B_3 times the expectation given B_3 . So this will satisfy the total expectation theorem.

So the probability of B_1 is going to be exactly $2/3$. It's simply the probability of a taxi waiting for A_1 . The expected value of X -- well, when B_1 happens, X is going to be exactly equal to 0. So the expected value is also going to be 0. The probability of B_2 happening is the probability of a taxi not being there times the probability of a taxi arriving between 0 and 5. It's going to be $1/3$ times $1/2$.

And the expected value of X given B_2 is going to be the expected value of this density. The expected value of this density is the midpoint between 0 and 5. And so it's going to be $5/2$. And the probability of B_3 is going to be $1/3$ times $1/2$. Finally, the expected value of X given B_3 . Well, when B_3 happens, X is going to be exactly equal to 5. So the expected value is also going to be 5. Now we're left with $5/12$ plus $5/6$, which is going to be $15/12$. And we can actually fill that in here so that we can clear up the board to do the other part.

Now we want to compute the CDF of X . Well, what is the CDF? Well, the CDF of X is going to be equal to the probability that the random variable X is less than or equal to some little x . It's a constant [INAUDIBLE].

Before we jump right in, let's try to understand what's the form of the CDF. And let's consider some interesting cases. You know that the random variable X , the waiting time, is going to be somewhere between 0 and 5, right? So let's consider what happens if little x is going to be less than 0. That's basically saying, what's the probability of the random variable X being less than some number that's less than 0? Waiting time can't be negative, so the probability of this is going to be 0.

Now, what if X is between equaling 0 and strictly less than 5? In that case, either X can fall between 0 and 5 according to this case, in the case of B_2 , or X can be exactly equal to 0. It's not clear. So let's do that later. Let's fill that in later.

What about if x is greater than or equal to 5? Little x , right? That's the probability that the random variable X is less than some number that's bigger than or equal to 5. The waiting time X , the random variable, is definitely going to be less than or equal to 5, so the probability of this is going to be 1.

So now this case. How do we do it? Well, let's try to use a similar kind of approach that we did for the expected value and use the total probability theorem in this case. So let's try to review

this. First of all, let's assume that this is true, that little x is between 0 and 5, including 0. And let's use the total probability theorem, and use the partitions B_1 , B_2 , and B_3 .

So what's the probability of B_1 ? It's the probability that A_1 catches waiting taxi, which happens with probability $2/3$. What's the probability that the random variable X , which is less than or equal to little x under this condition, when B_1 happens? Well, if B_1 happens, then random variable X is going to be exactly equal to 0, right? So in that case, it's definitely going to be less than or equal to any value of x , including 0. So the probability will be 1.

What's the probability that B_2 happens now? The probability that B_2 happens is $1/3$ times $1/2$, as we did before. And the probability that the random variable X is less than or equal to little x when B_2 happens. Well, if B_2 happens, this is your density. And this is our condition. And so x is going to be somewhere in between these spots.

And we'd like to compute what's the probably that random variable X is less than or equal to little x . So we want this area. And that area is going to have height of $1/5$ and width of x . And so the area's going to be $1/5$ times x .

And finally, the probability that B_3 happens is going to be $1/3$ times $1/2$ again times the probability that the random variable X is less than or equal to little x given B_3 . Well, when B_3 happens, X is going to be exactly 5 as a random variable. But little x , you know-- we're assuming in this condition-- is going to be between 0 and 5, but strictly less than 5. So there's no way that if the random variable X is 5 and this is strictly less than 5, this is going to be true. And so that probability will be 0.

So we're now left with $2/3$ plus $1/30$. And now we can fill this in. $2/3$ plus $1/30 x$. And this is our CDF.

So now we've finished the problem, computed the expected value here and then the CDF here, and this was a great illustration of how you would do so for a mixed random variable.

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