Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041SC Probabilistic Systems Analysis and Applied Probability

Lecture 12 Bonus Video Solution

Problem 27.* We toss n times a biased coin whose probability of heads, denoted by q, is the value of a random variable Q with given mean μ and positive variance σ^2. Let X_i be a Bernoulli random variable that models the outcome of the ith toss (i.e., $X_i = 1$ if the ith toss is a head). We assume that $X_1, \ldots, X_n$ are conditionally independent, given $Q = q$. Let X be the number of heads obtained in the n tosses.

(a) Use the law of iterated expectations to find $\mathbf{E}[X_i]$ and $\mathbf{E}[X]$.

(b) Find $\text{cov}(X_i, X_j)$. Are $X_1, \ldots, X_n$ independent?

(c) Use the law of total variance to find $\text{var}(X)$. Verify your answer using the covariance result of part (b).

Solution. (a) We have, from the law of iterated expectations and the fact $\mathbf{E}[X_i \,|\, Q] = Q$,

$$\mathbf{E}[X_i] = \mathbf{E}\big[\mathbf{E}[X_i \,|\, Q]\big] = \mathbf{E}[Q] = \mu.$$

Since $X = X_1 + \cdots + X_n$, it follows that

$$\mathbf{E}[X] = \mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n] = n\mu.$$

(b) We have, for $i = j$, using the conditional independence assumption,

$$\mathbf{E}[X_i X_j \,|\, Q] = \mathbf{E}[X_i \,|\, Q]\,\mathbf{E}[X_j \,|\, Q] = Q^2,$$

and

$$\mathbf{E}[X_i X_j] = \mathbf{E}\big[\mathbf{E}[X_i X_j \,|\, Q]\big] = \mathbf{E}[Q^2].$$

Thus,

$$\text{cov}(X_i, X_j) = \mathbf{E}[X_i X_j] - \mathbf{E}[X_i]\,\mathbf{E}[X_j] = \mathbf{E}[Q^2] - \mu^2 = \sigma^2.$$

Since $\text{cov}(X_i, X_j) > 0$, $X_1, \ldots, X_n$ are not independent.

Also, for $i = j$, using the observation that $X_i^2 = X_i$,

$$\begin{aligned}\text{var}(X_i) &= \mathbf{E}[X_i^2] - \big(\mathbf{E}[X_i]\big)^2 \\ &= \mathbf{E}[X_i] - \big(\mathbf{E}[X_i]\big)^2 \\ &= \mu - \mu^2.\end{aligned}$$

(c) Using the law of total variance, and the conditional independence of $X_1, \ldots, X_n$, we have

$$\begin{aligned}
\text{var}(X) &= \mathbf{E}\ \text{var}(X \mid Q)\ \ + \text{var}\ \ \mathbf{E}[X \mid Q] \\
&= \mathbf{E}\ \text{var}(X_1 + \cdots + X_n \mid Q)\ \ + \text{var}\ \ \mathbf{E}[X_1 + \cdots + X_n \mid Q] \\
&= \mathbf{E}\ nQ(1 - Q)\ \ + \text{var}(nQ) \\
&= n\mathbf{E}[Q - Q^2] + n^2 \text{var}(Q) \\
&= n(\mu - \mu^2 - \sigma^2) + n^2\sigma^2 \\
&= n(\mu - \mu^2) + n(n-1)\sigma^2.
\end{aligned}$$

To verify the result using the covariance formulas of part (b), we write

$$\begin{aligned}
\text{var}(X) &= \text{var}(X_1 + \cdots + X_n) \\
&= \sum_{i=1}^{n} \text{var}(X_i) + \sum_{\{(i,j) \mid i \neq j\}} \text{cov}(X_i, X_j) \\
&= n\text{var}(X_1) + n(n-1)\text{cov}(X_1, X_2) \\
&= n(\mu - \mu^2) + n(n-1)\sigma^2.
\end{aligned}$$