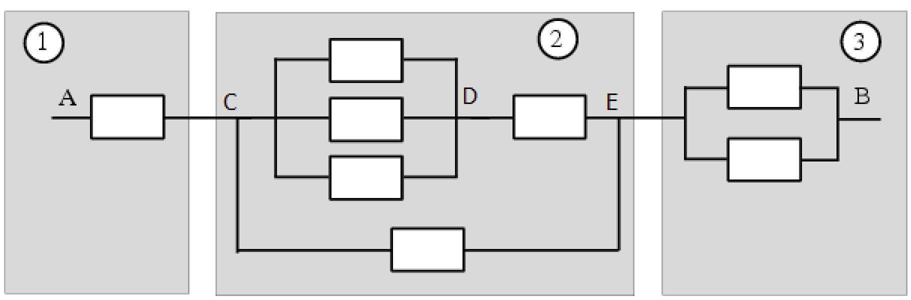
Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Tutorial 1 Solutions
September 16/17, 2010

1. If $A \subset B$, then $\mathbf{P}(B \cap A) = \mathbf{P}(A)$ But we know that in order for A and B to be independent, $\mathbf{P}(B \cap A) = \mathbf{P}(A)\mathbf{P}(B)$. Therefore, A and B are independent if and only if $\mathbf{P}(B) = 1$ or $\mathbf{P}(A) = 0$. This could happen, for example, if B is the universe or if A is empty.

2. This problem is similar in nature to Example 1.24, page 40. In order to compute the success probability of individual sub-systems, we make use of the following two properties, derived in that example:

   - If a *serial* sub-system contains m components with success probabilities $p_1, p_2 ... p_m$, then the probability of success of the entire sub-system is given by
   $$\mathbf{P}(\text{whole system succeeds}) = p_1 p_2 p_3 ... p_m$$
   - If a *parallel* sub-system contains m components with success probabilities $p_1, p_2 ... p_m$, then the probability of success of the entire sub-system is given by
   $$\mathbf{P}(\text{whole system succeeds}) = 1 - (1 - p_1)(1 - p_2)(1 - p_3)...(1 - p_m)$$



Let $\mathbf{P}(X \to Y)$ denote the probability of a successful connection between node X and Y. Then,

$$\begin{aligned}
\mathbf{P}(A \to B) &= \mathbf{P}(A \to C)\mathbf{P}(C \to E)\mathbf{P}(E \to B) \text{ (since they are in series)} \\
\mathbf{P}(A \to C) &= p \\
\mathbf{P}(C \to E) &= 1 - (1-p)\,(1 - \mathbf{P}(C \to D)\mathbf{P}(D \to E)) \\
\mathbf{P}(E \to B) &= 1 - (1-p)^2
\end{aligned}$$

The probabilities $\mathbf{P}(C \to D)$, $\mathbf{P}(D \to E)$ can be similarly computed as

$$\begin{aligned}
\mathbf{P}(C \to D) &= 1 - (1-p)^3 \\
\mathbf{P}(D \to E) &= p
\end{aligned}$$

The probability of success of the entire system can be obtained by substituting the subsystem success probabilities:

$$\mathbf{P}(A \to B) = p\,(1 - (1-p)(1 - (1 - (1-p)^3)p)\,(1 - (1-p)^2).$$

3. **The Chess Problem.**

(a) i. $\mathbf{P}(\text{2nd Rnd Req}) = (0.6)^2 + (0.4)^2 = 0.52$

ii. $\mathbf{P}(\text{Bo Wins 1st Rnd}) = (0.6)^2 = 0.36$

iii. $\mathbf{P}(\text{Al Champ}) = 1 - \mathbf{P}(\text{Bo Champ}) - \mathbf{P}(\text{Ci Champ})$
$= 1 - (0.6)^2 * (0.5)^2 - (0.4)^2 * (0.3)^2 = 0.8956$

(b) i. $\mathbf{P}(\text{Bo Challenger}|\text{2nd Rnd Req}) = \frac{(0.6)^2}{0.52} = \frac{0.36}{0.52} = 0.6923$

ii. $\mathbf{P}(\text{Al Champ}|\text{2nd Rnd Req})$
$= \mathbf{P}(\text{Al Champ}|\text{Bo Challenger, 2nd Rnd Req}) \times \mathbf{P}(\text{Bo Challenger}|\text{2nd Rnd Req})$
$+ \ \mathbf{P}(\text{Al Champ}|\text{Ci Challenger, 2nd Rnd Req}) \times \mathbf{P}(\text{Ci Challenger}|\text{2nd Rnd Req})$
$= (1 - (0.5)^2) \times 0.6923 + (1 - (0.3)^2) \times 0.3077$
$= 0.7992$

(c) $\mathbf{P}((\text{Bo Challenger})|\{(\text{2nd Rnd Req}) \cap (\text{One Game})\}) = \frac{(0.6)^2 * (0.5)}{(0.6)^2 * (0.5) + (0.4)^2 * (0.7)}$
$= \frac{(0.6)^2(0.5)}{0.2920} = 0.6164$