LECTURE 3

- **Readings:** Section 1.5
- Review
- Independence of two events
- Independence of a collection of events

Review

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}, \qquad \text{assuming } \mathbf{P}(B) > 0$$

- Multiplication rule:

$$\mathbf{P}(A \cap B) = \mathbf{P}(B) \cdot \mathbf{P}(A \mid B) = \mathbf{P}(A) \cdot \mathbf{P}(B \mid A)$$

- Total probability theorem:

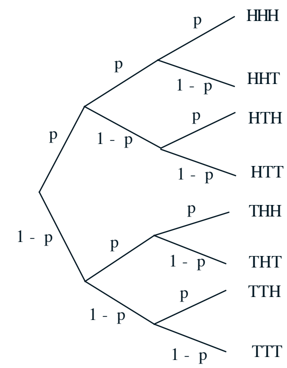
$$\mathbf{P}(B) = \mathbf{P}(A)\mathbf{P}(B \mid A) + \mathbf{P}(A^c)\mathbf{P}(B \mid A^c)$$

- Bayes rule:

$$\mathbf{P}(A_i \mid B) = \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\mathbf{P}(B)}$$

Models based on conditional probabilities

- 3 tosses of a biased coin:
  $\mathbf{P}(H) = p$, $\mathbf{P}(T) = 1 - p$



$\mathbf{P}(THT) =$

$\mathbf{P}(1 \text{ head}) =$

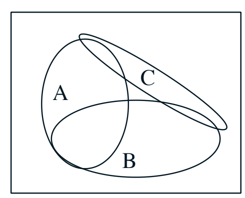
$\mathbf{P}(\text{first toss is H} \mid 1 \text{ head}) =$

Independence of two events

- **"Defn:"** $\quad \mathbf{P}(B \mid A) = \mathbf{P}(B)$
  - "occurrence of A provides no information about B's occurrence"
- Recall that $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B \mid A)$
- **Defn:** $\quad \mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$
- Symmetric with respect to A and B
  - applies even if $\mathbf{P}(A) = 0$
  - implies $\mathbf{P}(A \mid B) = \mathbf{P}(A)$

Conditioning may affect independence

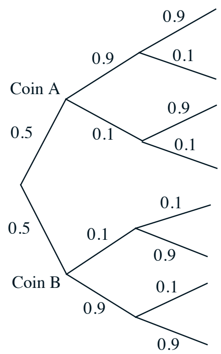
- Conditional independence, given C, is defined as independence under probability law $\mathbf{P}(\cdot \mid C)$
- Assume A and B are independent



- If we are told that C occurred, are A and B independent?

Conditioning may affect independence

- Two unfair coins, A and B:
  $\mathbf{P}(H \mid \text{coin } A) = 0.9$, $\mathbf{P}(H \mid \text{coin } B) = 0.1$
  choose either coin with equal probability



- Once we know it is coin A, are tosses independent?
- If we do not know which coin it is, are tosses independent?
  - Compare:
    $\mathbf{P}(\text{toss } 11 = H)$
    $\mathbf{P}(\text{toss } 11 = H \mid \text{first 10 tosses are heads})$

Independence of a collection of events

- Intuitive definition:
  Information on some of the events tells us nothing about probabilities related to the remaining events
  - E.g.:

$$\mathbf{P}(A_1 \cap (A_2^c \cup A_3) \mid A_5 \cap A_6^c) = \mathbf{P}(A_1 \cap (A_2^c \cup A_3))$$

- Mathematical definition:
  Events $A_1, A_2, \ldots, A_n$
  are called **independent** if:

$$\mathbf{P}(A_i \cap A_j \cap \cdots \cap A_q) = \mathbf{P}(A_i)\mathbf{P}(A_j)\cdots\mathbf{P}(A_q)$$

  for any distinct indices $i, j, \ldots, q$,
  (chosen from $\{1, \ldots, n\}$)

Independence vs. pairwise independence

- Two independent fair coin tosses
  - A: First toss is H
  - B: Second toss is H
  - $\mathbf{P}(A) = \mathbf{P}(B) = 1/2$

HH	HT
TH	TT

  - C: First and second toss give same result
  - $\mathbf{P}(C) =$
  - $\mathbf{P}(C \cap A) =$
  - $\mathbf{P}(A \cap B \cap C) =$
  - $\mathbf{P}(C \mid A \cap B) =$
- Pairwise independence **does not** imply independence

The king's sibling

- The king comes from a family of two children. What is the probability that his sibling is female?