

## 6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Recitation: Flipping a Coin a Random Number of Times

In this problem, we're looking at a two stage process in which the first stage, we roll a fair die which has four faces to obtain a number  $N$ , where  $N$  belongs to the set 0, 1, 2, and 3 with equal probability. Now, given the result of the die roll,  $N$  will toss a fair coin  $N$  times in getting  $K$  heads from the coin tosses.

For instance, if from the first die roll, we get  $N$  equal to 3, then we'll toss a coin 3 times. Let's say the outcome is heads, heads, and tails. And that will give us  $K$  equal to 2.

For part A, we're asked to compute the PMF for  $N$ , which is a result of the first die roll. Now, since we had assumed the die roll was uniformly distributed in the set in the set 0, 1, 2, and 3, we have that the chance of  $N$  being equal to any little  $n$  is equal to  $1/4$  if  $n$  is in the set 0, 1, 2, 3, and 0 otherwise. If we were to plot this in a figure, we'll have the following plot.

For part B, things are getting a little more complicated. This time, we want to compute the joint PMF between  $N$  and  $K$  for  $N$  equal to little  $n$  and  $K$  equal to little  $k$ . What we'll do first is to use the law of conditional probability to break the joint probability into the product of probability of  $K$  is equal to little  $k$  conditional on  $N$  is equal to little  $n$ , multiply by the probability that  $N$  is equal to little  $n$ .

Now, the second term right here is simply the PMF of  $N$ , which will be computed earlier. So this gives us  $1/4$  times probability  $K$  equal to little  $k$ ,  $N$  equal to little  $n$ , for all  $N$  in the set 0, 1, 2, and 3. Now, clearly if  $N$  is not one of those four values, this whole event couldn't have happened in the first place, and hence will have  $P$  and  $K$  equal to 0.

We'll now go over all the cases for little  $n$  in this expression right here. The first case is the simplest. If we assume that little  $n$  is equal to 0, that means the die roll was 0, and hence we're not tossing any coins afterwards. And this implies that we must have  $K$  is equal to 0, which, mathematically speaking, is equivalent to saying probability of  $K$  equal to 0 conditional on  $N$  equal to 0 is 1. And  $K$  being any other value conditional  $N$  equal to 0 is 0. So we're done with the case that little  $n$  is equal to 0.

Now, let's say little  $n$  is in the set 1, 2, and 3. In this case, we want to notice that after having observed the value of  $N$ , all the coin tosses for  $N$  times are conditionally independent from each other. What this means is now the total number of heads in the subsequent coin toss is equal in distribution to a binomial random variable with parameter  $n$  and  $1/2$ . And here says the number of trials is  $n$ , and  $1/2$  is because the coin is fair. And the reason it is a binomial random variable, again, is because the coin tosses are independent conditional on the outcome of the die roll.

And now we're done, since we know what the binomial distribution looks like given parameter  $n$  and  $1/2$ . And we'll simply substitute based on the case of  $n$  the conditional distribution of  $K$  back into the product we had earlier, which in turn will give us the joint PMF.

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This table summarizes the PMF we were computing earlier.  $P$  of  $N$ ,  $K$ , little  $n$ , and little  $k$ . Now, as we saw before, if  $n$  equal to 0, the only possibility for  $k$  is equal to 0. And this is scaled by the probability of  $n$  equal to 0, which is  $1/4$ . For any other values of  $n$ , we see that the distribution of  $k$ , conditional  $n$ , is the same as a binomial random variable with  $n$  trials. And again, every entry here is scaled by  $1/4$ . And this completes part B.

In part C, we're asked for the conditional PMF of  $K$  conditioning on the value of  $N$  being equal to 2. Now, as we discussed in part B, when  $N$  is equal to 2, we're essentially flipping a fair coin twice, and this should give us the same distribution as a binomial random variable with parameter 2 and  $1/2$ . Now, 2 is the number of flips, and  $1/2$  is the chance of seeing a head in each flip. And that gives us the following distribution.

But there's another way to see this. It's to write  $P$   $K$  given  $N$ , little  $k$ , and you go to 2 by using the law of conditional probability as  $P$   $K$ ,  $N$ , the joint PMF,  $k$   $n=2$ , divided by the probability that  $N$  is equal to 2.

Now, we know that probability  $n$  equal to 2 is simply  $1/4$ , so this gives us 4 times the joint density  $K$ ,  $N$ ,  $k$ , 2. In other words, in order to arrive at the distribution right here, [INAUDIBLE] to go back to the table we had earlier and look at the row where  $n$  is equal to 2 and multiply each number by 4.

Finally, in part D, we're asked for the conditional distribution of  $N$ , write as  $P$   $N$ , given  $K$  of  $N$  equal to little  $n$  conditional on  $K$  is equal to 2. Again, we'll apply the formula for conditional probability. This is equal to the joint PMF evaluated at  $n$  and 2 divided by the probability of  $K$  being equal to 2.

Since we have computed the entire table of the joint PMF, this shouldn't be too difficult. In particular, for the denominator, the probability that  $k$  is ever equal to 2, we just look at the column right here. So the entries in this column shows all the cases where  $k$  can be equal to 2. And in fact, we can see that  $k$  can be equal to 2 only if  $n$  is equal to 2 or 3. Clearly, if you toss the coin fewer than 2 times, there's no chance that we'll get 2 heads. So to get this probability right here, we'll add up the number in these two cells. So we get  $P$   $N$ ,  $K$ , little  $n$ , and 2 divided by  $1/16$  plus  $3/32$ .

Now, the numerator, again, can be read off from the table right here. In particular, this tells us that there are only two possibilities. Either  $n$  is equal to 2 or  $n$  equal to 3. When  $n$  is equal to 2, we know this quantity gives us  $1/16$  reading off this cell divided by  $1/16$  plus  $3/32$  for  $n$  equal to 2. And the remaining probability goes to the case where  $n$  is equal to 3.

So this is 3 divided by 32,  $1/16$  plus  $3/32$ , which simplifies to  $2/5$  and  $3/5$ . And this distribution gives us the following plot. And this completes our problem.