MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

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1. See textbook pg. 399

2. (a) $N = 200,000$.
   (b) $N = 100,000$.

3. Let us fix some $\epsilon > 0$. We will show that $P(Y_n \geq 0.5 + \epsilon)$ converges to 0. By symmetry, this will imply that $P(Y_n \leq 0.5 - \epsilon)$ also converges to zero, and it will follow that Y_n converges to 0.5, in probability.

   For the event $\{Y_n \geq 0.5 + \epsilon\}$ to occur, we must have at least $n+1$ of the random variables $X_1, X_2, \ldots, X_{2n+1}$ to have a value of $0.5 + \epsilon$ or larger. Let Z_i be a Bernoulli random variable which is equal to 1 if and only if $X_i \geq 0.5 + \epsilon$:

$$Z_i = \begin{cases} 1 & \text{if } X_i \geq 0.5 + \epsilon \\ 0 & \text{otherwise} \end{cases}$$

   $\{Z_1, Z_2, ...\}$ are i.i.d random variables and $E[Z_i] = P(Z_i = 1) = P(X_i \geq 0.5 + \epsilon) = 0.5 - \epsilon$.

   Hence, for the event $\{Y_n \geq 0.5 + \epsilon\}$ to occur, we must have at least $n+1$ of the $\{Z_i\}$ to take value 1,

$$\begin{aligned} P(Y_n \geq 0.5 + \epsilon) &= P(\sum_{i=1}^{2n+1} Z_i \geq n+1) \\ &= P(\frac{\sum_{i=1}^{2n+1} Z_i}{2n+1} \geq \frac{n+1}{2n+1}) \\ &= P(\frac{\sum_{i=1}^{2n+1} Z_i}{2n+1} \geq 0.5 + \frac{1}{2(2n+1)}) \\ &\leq P(\frac{\sum_{i=1}^{2n+1} Z_i}{2n+1} \geq 0.5) \end{aligned}$$

   Note that $P(Z_i = 1) = 0.5 - \epsilon$. By the weak law of large numbers, the sequence $(Z_1 + \cdots + Z_{2n+1})/(2n+1)$ converges to $0.5 - \epsilon$. To show that $P\left(\frac{Z_1 + \cdots + Z_{2n+1}}{2n+1} \geq 0.5\right)$ converges to zero, we need to show that for any given $\epsilon > 0$, there exists N such that for all $n > N$, $P\left(\frac{Z_1 + \cdots + Z_{2n+1}}{2n+1} \geq 0.5\right) < \epsilon$. The fact that the sequence $(Z_1 + \cdots + Z_{2n+1})/(2n+1)$ converges to $0.5 - \epsilon$ ensures the existence of such N. Since $P(Y_n \geq 0.5 + \epsilon)$ is bounded by $P(\frac{\sum_{i=1}^{2n+1} Z_i}{2n+1} \geq 0.5)$, it also converges to zero.