LECTURE 23

- Readings: Section 7.4, 7.5

Lecture outline

- Proof of the central limit theorem
- Approximating binomial distributions

CLT Review

- $X_1, \cdots, X_n$ i.i.d. finite variance σ^2
- $S_n = X_1 + \cdots + X_n$ variance $n\sigma^2$
- $Z_n = \dfrac{(X_1 + \cdots + X_n) - n\mathbf{E}[X]}{\sigma\sqrt{n}}$
- Z standard normal (zero mean, unit variance)
- **CLT**: For every c : $\mathbf{P}(Z_n \leq c) \to \mathbf{P}(Z \leq c) = \Phi(c)$
- Normal approximation:
  - Treat S_n as if normal.

“Proof” of the CLT

- Assume for simplicity $\mathbf{E}[X] = 0, \sigma = 1$
- Need to show that $Z_n = \dfrac{X_1 + \cdots + X_n}{\sqrt{n}}$

  converges to standard normal.

- We have:

$$M_{Z_n}(s) = \mathbf{E}[e^{sZ_n}] = \mathbf{E}\left[e^{(s/\sqrt{n})(X_1+\cdots+X_n)}\right]$$

$$\mathbf{E}[e^{sX/\sqrt{n}}] \approx 1 + \frac{s}{\sqrt{n}}\mathbf{E}[X] + \frac{s^2}{2n}\mathbf{E}[X^2]$$

$$M_{Z_n}(s) = (\mathbf{E}[e^{sX/\sqrt{n}}])^n \approx \left(1 + \frac{s^2}{2n}\right)^n \longrightarrow e^{s^2/2}$$

which is the transform of the standard normal.

Apply to Binomial

- Fix p , where $0 < p < 1$
- X_i: Bernoulli(p)
- $S_n = X_1 + \cdots + X_n$: Binomial(n, p)
  - mean np , variance $np(1-p)$
- $\frac{S_n - np}{\sqrt{np(1-p)}} \longrightarrow$ standard normal CDF

Example

- $n = 36, p = 0.5$; find $\mathbf{P}(S_n \leq 21)$

- $\mathbf{P}(S_n \leq 21) = \Phi(\frac{21-18}{3}) = \Phi(1) = .9413$

- Exact answer:

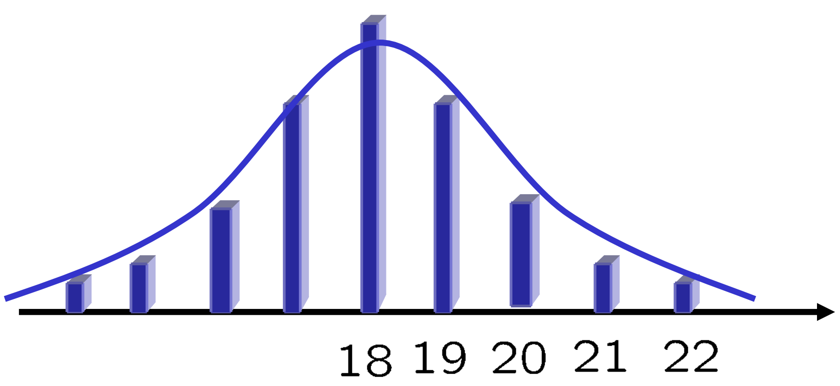
$$\sum_{k=0}^{21} \binom{36}{k} \left(\frac{1}{2}\right)^{36} = 0.8785$$

The ½ correction for binomial approximation

- $\mathrm{P}(S_n \leq 21) = \mathrm{P}(S_n < 22)$

  because S_n is integer.

- Compromise: consider $\mathrm{P}(S_n \leq 21.5)$



De Moivre-Laplace CLT (for binomial)

- When the ½ correction is used, CLT can also approximate the binomial PMF (not just the CDF).

$$\mathbf{P}(S_n = 19) = \mathbf{P}(18.5 \le S_n \le 19.5)$$

$$18.5 \le S_n \le 19.5 \quad \Longleftrightarrow$$

$$\frac{18.5 - 18}{3} \le \frac{S_n - 18}{3} \le \frac{19.5 - 18}{3} \quad \Longleftrightarrow$$

$$0.17 \le Z_n \le 0.5$$

$$\begin{aligned} \mathbf{P}(S_n = 19) &\approx \mathbf{P}(0.17 \le Z \le 0.5) \\ &= \mathbf{P}(Z \le 0.5) - \mathbf{P}(Z \le 0.17) \\ &= 0.6915 - 0.5675 \\ &= 0.124 \end{aligned}$$

- Exact answer: $\binom{36}{19}\left(\frac{1}{2}\right)^{36} = 0.1251$

Poisson vs. normal approximations of the binomial

- Binomial (n, p)
  - p fixed, $n \longrightarrow \infty$: normal
  - np fixed, $n \longrightarrow \infty,\ p \longrightarrow 0$: Poisson
- $p = 1/100,\ n = 100$: Poisson
- $p = 1/10,\ n = 500$: normal