LECTURE 21

- Readings: Section 6.4

Lecture outline

- Markov Processes – III
  - Review of steady-state behavior
  - Queuing applications
  - Calculating absorption probabilities
  - Calculating expected time to absorption

Review

- Assume a single class of recurrent states, aperiodic. Then,

$$\lim_{n \to \infty} r_{ij}(n) = \pi_j$$

where π_j does not depend on the initial conditions

$$\lim_{n \to \infty} \mathbf{P}(X_n = j | X_0) = \pi_j$$

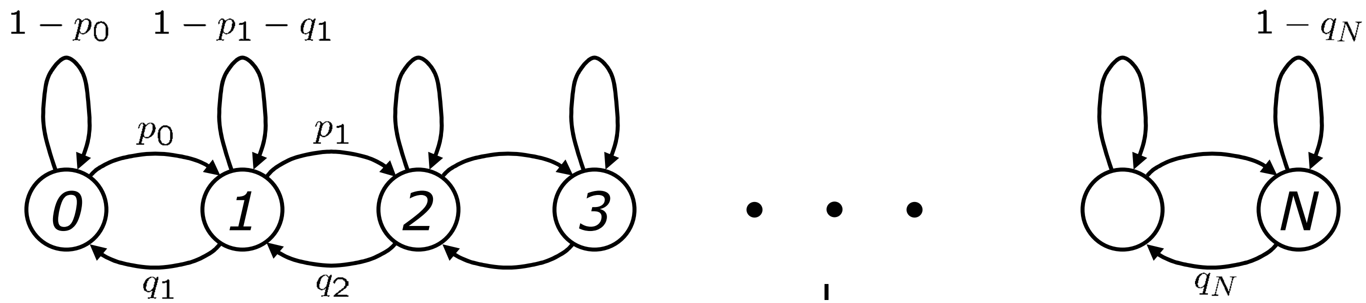
- $\pi_1, \cdots, \pi_m$ can be found as the unique solution of the balance equations:

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj}$$

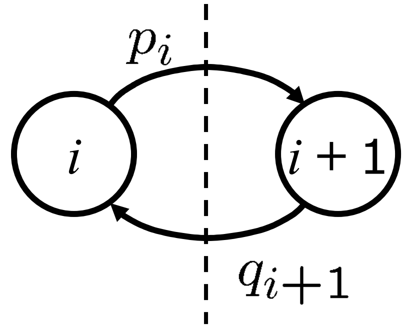
together with $\sum_{k=1}^{m} \pi_j = 1$

Birth-Death Process

- General case:



- Locally, we have:



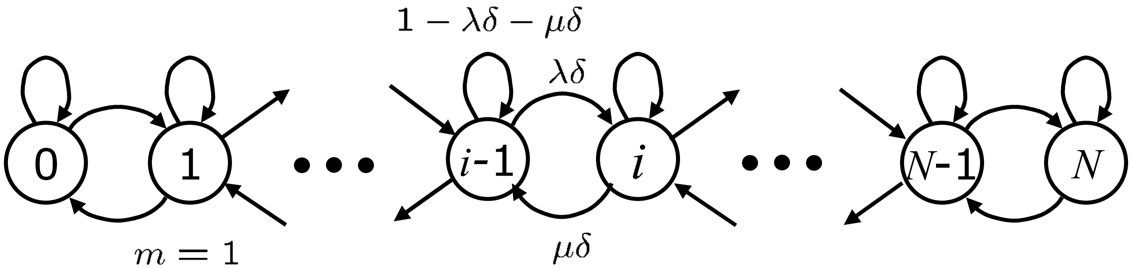
- Balance equations: $\pi_i p_i = \pi_{i+1} q_{i+1}$

- Why? (More powerful, e.g. queues, etc.)

M/M/1 Queue (1)

- Poisson **arrivals** with rate λ
- Exponential **service time** with rate μ
- $m = 1$ **server**
- Maximum **capacity** of the system = N

- Discrete time intervals of (small) length δ :



- Balance equations: $\lambda \pi_{i-1} = \mu \pi_i \quad i \leq N$
- Identical solution to the the random walk problem.

M/M/1 Queue (2)

- Define: $\rho = \dfrac{\lambda}{\mu}$
- Then: $\pi_{i+1} = \pi_i \dfrac{\lambda}{\mu} = \pi_i \rho$

$$\pi_i = \pi_0 \rho^i, \qquad i = 0, 1, \cdots, m$$

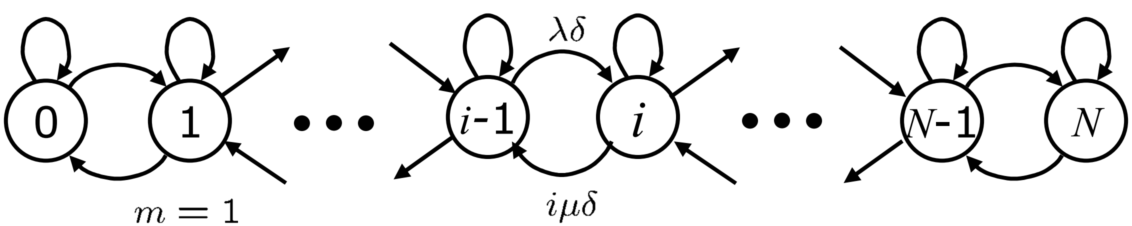
- To get π_0, use: $\sum_j \pi_j = 1$

$$\pi_0 = \frac{1}{1 + \rho + \cdots + \rho^m} = \frac{1 - \rho}{1 - \rho^{m+1}}$$

- Consider 2 cases!

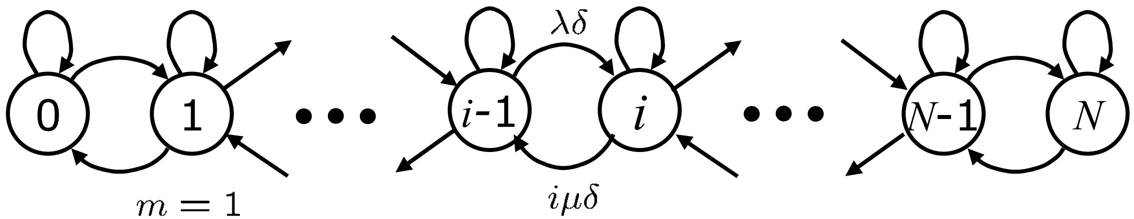
The Phone Company Problem (1)

- Poisson arrivals (**calls**) with rate λ
- Exponential service time (**call duration**), rate μ
- $m = N$ servers (**number of lines**)
- Maximum capacity of the system = N
- Discrete time intervals of (small) length δ :



- Balance equations: $\lambda \pi_{i-1} = i\mu\pi_i$
- Solve to get: $\pi_i = \pi_0 \frac{\lambda^i}{\mu^i i!}$ $\quad \pi_0 = 1 / \sum_{i=0}^{N} \frac{\lambda^i}{\mu^i i!}$

The Phone Company Problem (2)



- Balance equations: $\lambda \pi_{i-1} = i \mu \pi_i$
- Solution: $\pi_i = \pi_0 \frac{\lambda^i}{\mu^i i!}$ $\quad \pi_0 = 1 / \sum_{i=0}^{N} \frac{\lambda^i}{\mu^i i!}$
- Consider the limiting behavior as $N \to \infty$.

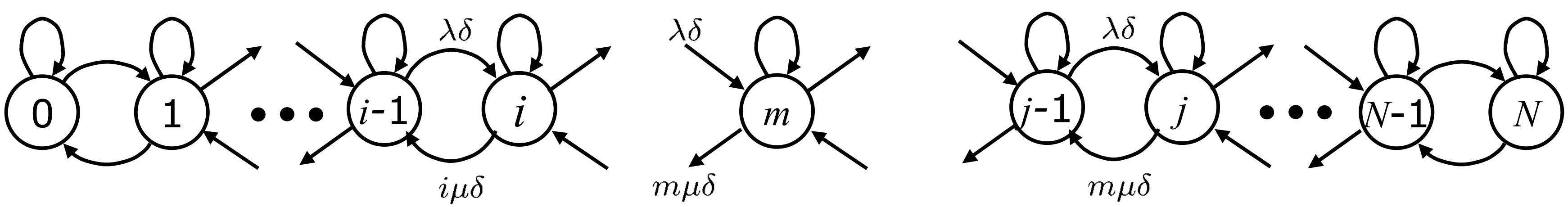
$$\pi_0 = \lim_{N \to \infty} 1 / \sum_{i=0}^{N} \frac{\lambda^i}{\mu^i i!} = e^{-\rho}$$

- Therefore: $\pi_i = e^{-\rho} \frac{\lambda^i}{\mu^i i!} = e^{-\rho} \frac{\rho^i}{i!}$ (Poisson)

M/M/m Queue

- Poisson **arrivals** with rate λ
- Exponential **service time** with rate μ
- m **servers**
- Maximum **capacity** of the system = N

- Discrete time intervals of (small) length δ :



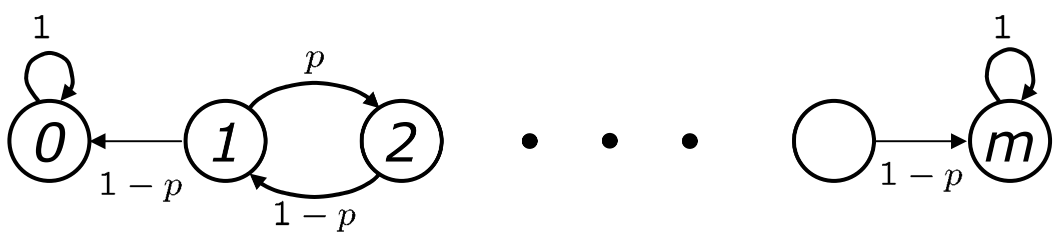
- Balance equations:

$$\lambda\pi_{i-1} = i\mu\pi_i \quad i \leq m$$

$$\lambda\pi_{i-1} = m\mu\pi_i \quad i > m$$

Gambler's Ruin (1)

- Each round, Charles Barkley wins 1 thousand dollars with probability p and looses 1 thousand dollars with probability $1 - p$
- Casino capital is equal to m
- He claims he does not have a gambling problem!



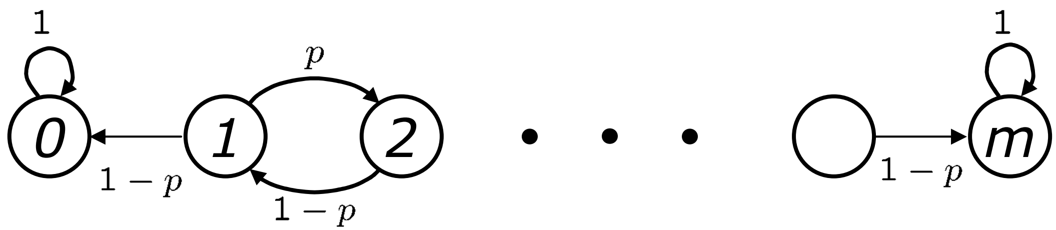
- Both 0 and m are absorbing!

Calculating Absorption Probabilities

- Each state is either transient or absorbing
- Let s be one absorbing state
- Definition: Let a_i be the probability that the state will eventually end up in s given that the chain starts in state i

- For $i = s,\ a_i = 1$
- For $i =$ other absorbing state, $a_i =$ 0
- For all other i : $a_i = \sum_j p_{ij} a_j$

Gambler's Ruin (2)



$$a_0 = 0$$

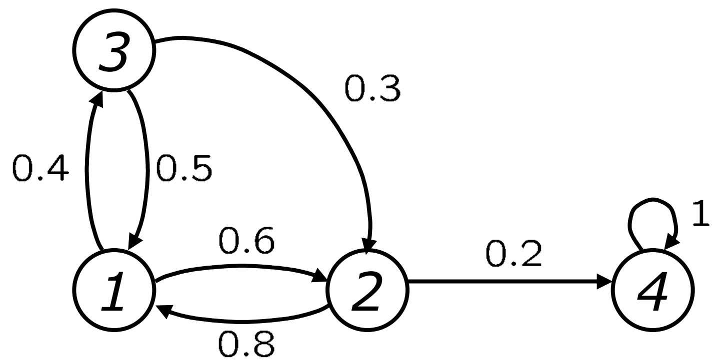
$$a_m = 1$$

$$a_i = (1-p)a_{i-1} + pa_i$$

$$\rho = \frac{1-p}{p}$$

$$a_i = \begin{cases} \frac{1-\rho^i}{1-\rho^m} & \text{if} \rho \neq 1 \\ \frac{i}{m} & \text{if} \rho = 1 \end{cases}$$

Expected Time to Absorption



- What is the expected number of transitions μ_i until the process reaches the absorbing state, given that the initial state is i ?
- $\mu_i = 0$ for $i = 4$
- For all other i : $\mu_i = 1 + \sum_j p_{ij}\mu_j$