LECTURE 19

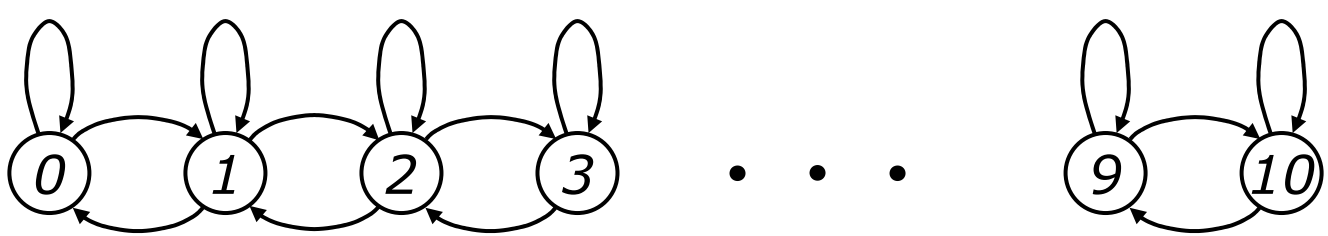
- Readings: Finish Section 5.2

Lecture outline

- Markov Processes – I
  - Checkout counter example.
  - Markov process: definition.
  - n-step transition probabilities.
  - Classification of states.

Example: **Checkout Counter**

- Discrete time $n = 0, 1, \cdots$
- Customer arrivals: Bernoulli(p)
  - Geometric interarrival times.
- Customer service times: Geometric(q)
- "State" X_n : number of customers at time n .



Finite State Markov Models

- X_n : state after n transitions
  - Belongs to a finite set, e.g. $\{1, \cdots, m\}$
  - X_0 is either given or random.
- **Markov Property / Assumption:**
  - Given the current state, the past does not matter.

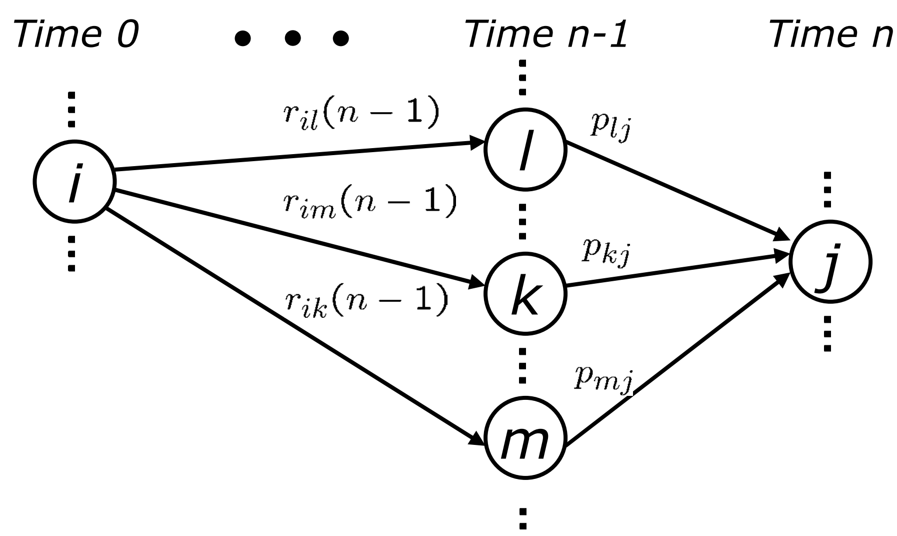
$$\begin{aligned} p_{ij} &= \mathrm{P}(X_{n+1} = j | X_n = i, X_{n-1}, \cdots, X_0) \\ &= \mathrm{P}(X_{n+1} = j | X_n = i) \end{aligned}$$

- Modeling steps:
  - Identify the possible states.
  - Mark the possible transitions.
  - Record the transition probabilities.

n-step Transition Probabilities

- State occupancy probabilities, given initial state i:

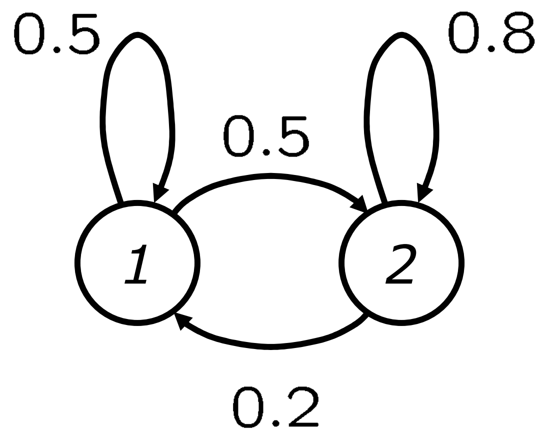
$$r_{ij}(n) = \mathbf{P}(X_n = j \mid X_0 = i)$$



- Key recursion: $$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}$$

- Random initial state: $$\mathbf{P}(X_n = j) = \sum_{i=1}^{m} \mathbf{P}(X_0 = i) r_{ij}(n)$$

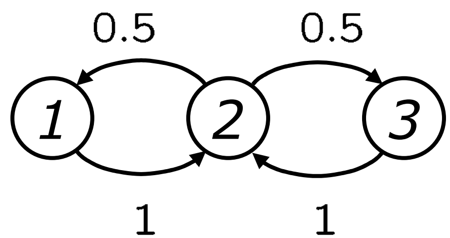
Example



	$n = 0$	$n = 1$	$n = 2$	$n = 2563$	$n = 2564$
$r_{11}(n)$	1	0.5	0.35		
$r_{12}(n)$	0	0.5	0.65		

Generic Question

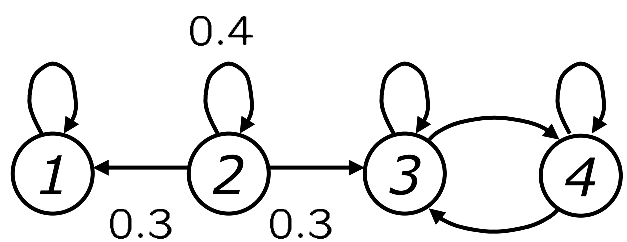
- Does $r_{ij}(n)$ converge to something?



n odd: $r_{22}(n) = 0$

n even: $r_{22}(n) = 1$

- Does the limit depend on the initial state?



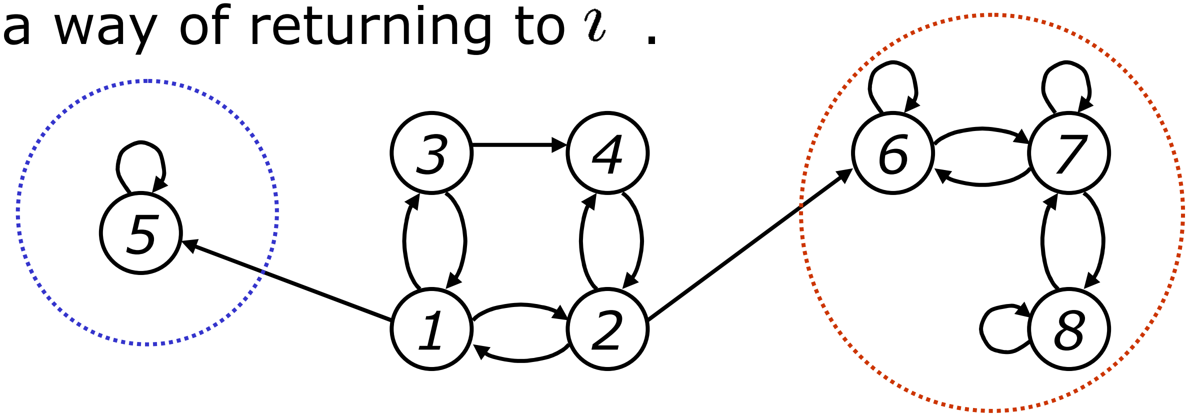
$r_{11}(n) = 1$

$r_{31}(n) = 0$

$$
\begin{aligned}
r_{21}(n) &= r_{21}(n-1) + p_{21} r_{22}(n-1) \\
&= r_{21}(n-1) + 0.3(0.4)^{n-1} \\
&= 0.3(1 + 0.4 + \cdots + 0.4^{n-1}) = 0.3\tfrac{1-0.4^n}{1-0.4} \\
&= 0.5
\end{aligned}
$$

Recurrent and Transient States

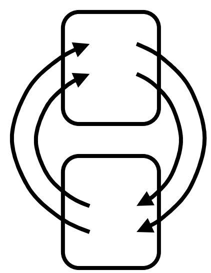
- State i is **recurrent** if:
  - Starting from i, and from wherever you can go, there is a way of returning to i .

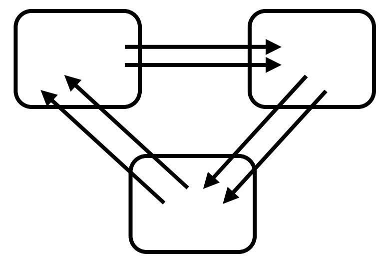


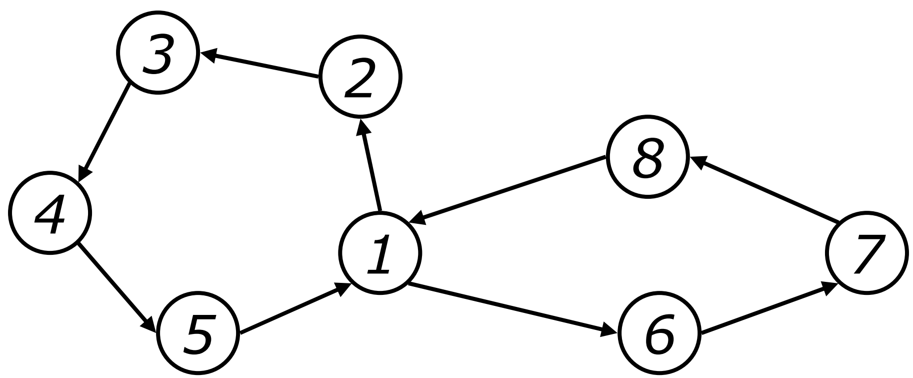
- If not recurrent, a state is called **transient**.
  - If i is transient then $\mathrm{P}(X_n = i) \rightarrow 0$ as $n \rightarrow \infty$.
  - State i is visited only a finite number of times.
- **Recurrent Class**:
  - Collection of recurrent states that "communicate" to each other, and to no other state.

Periodic States

- The states in a recurrent class are **periodic** if:
  - They can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.







- In this case, $r_{ii}(n)$ cannot converge.