LECTURE 14

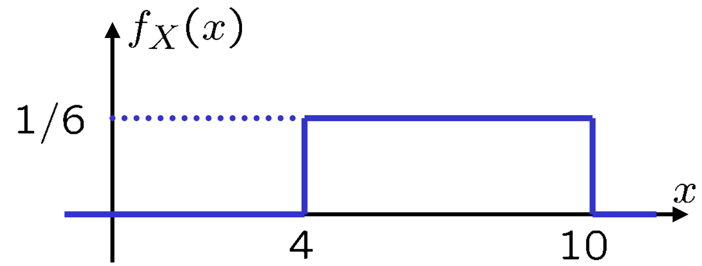
- Readings: Section 4.5, 4.6 (optional)

Lecture outline

- Prediction
  - Without and With Information.
- Covariance and Correlation
  - Definitions.
  - Variance of the Sum of Random Variables.

Prediction in the Absence of Information

- PDF of a random variable X is known.
- Example:



- **Prediction:**
  - What is the best guess c of the value of X ?
- "Best" = $\boxed{\text{minimize } \mathbf{E}\left[(X-c)^2\right]}$
- Solution: $\boxed{c = \mathbf{E}[X]}$
- Optimal (least) mean squared error:

$$\boxed{\mathbf{E}\left[(X-\mathbf{E}[X])^2\right] = \text{Var}(X)}$$

Prediction with Information

- Two random variables: X, Y
- We observe that $Y = y$
  - New universe: condition on $Y = y$.
- $\mathbf{E}\left[(X - c)^2 \middle| Y = y\right]$ is minimized by:

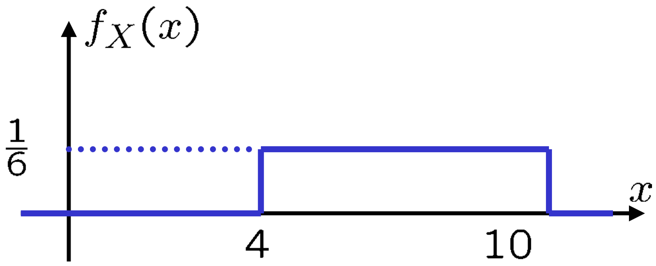
$$\boxed{c = \mathbf{E}[X|Y = y]}$$

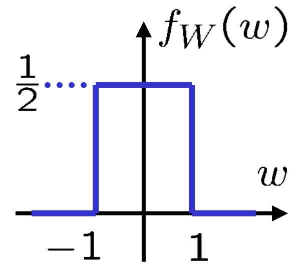
- View the predictor as a function $g(y)$.
- $\mathbf{E}[X|Y]$ minimizes $\mathbf{E}\left[(X - g(Y))^2\right]$

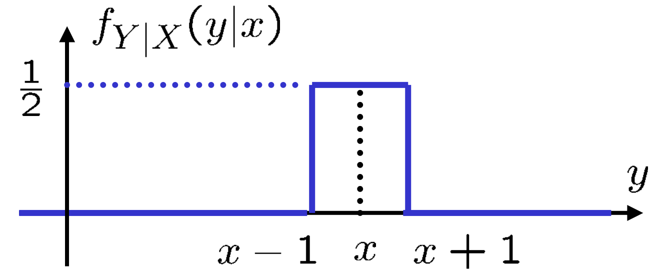
  over all predictors $g(\cdot)$.

Example

- $Y = X + W$, X and W are independent.

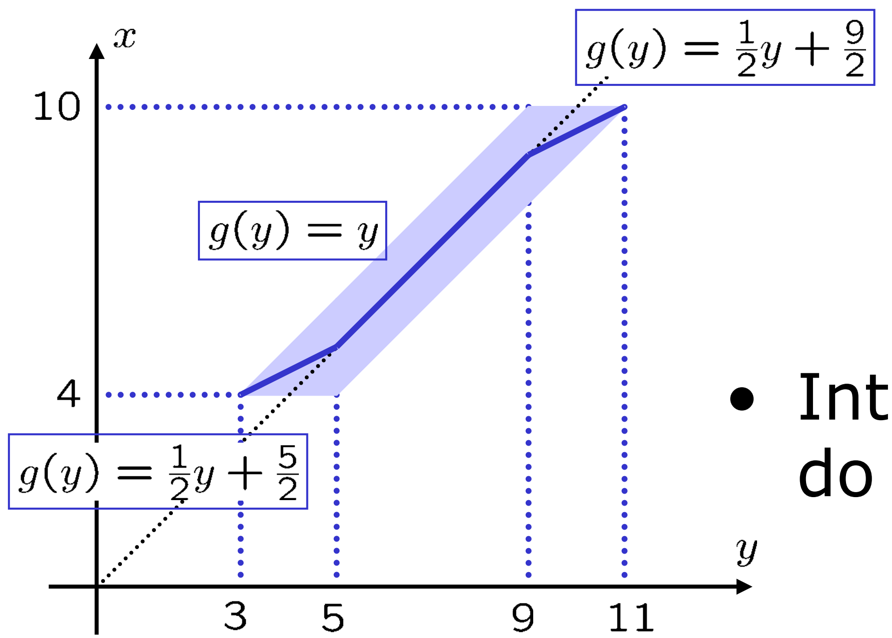


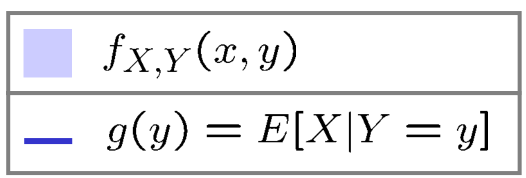




- Temptation: $g(y) = y$ (because of symmetry)

- But:





- Intuition: We can do better at edges.

Covariance and Correlation

- **Covariance**:

$$\text{Cov}(X,Y) = \mathbf{E}\left[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])\right]$$

- **Correlation**: (dimensionless version of covariance)

$$\rho = \mathbf{E}\left[\frac{(X - \mathbf{E}[X])}{\sigma_X}\frac{(Y - \mathbf{E}[Y])}{\sigma_Y}\right]$$

- Property: $-1 \le \rho \le 1$

- Independence $\Longrightarrow$ Zero covariance (uncorrelated)

  (The converse is not true!)

Variance of the Sum of r.v.s

- Recall, if X_1, X_2 independent, then:

$$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2)$$

- How about the dependent case?

- Let $\tilde{X}_1 = X_1 - \mathbf{E}[X_1]$ and $\tilde{X}_2 = X_2 - \mathbf{E}[X_2]$, then:

$$\begin{aligned}\text{Var}(X_1 + X_2) &= \mathbf{E}[(\tilde{X}_1 + \tilde{X}_2)^2] \\ &= \mathbf{E}[\tilde{X}_1^2] + \mathbf{E}[\tilde{X}_2^2] + 2\mathbf{E}[\tilde{X}_1 \tilde{X}_2] \\ &= \text{Var}(X_1) + \text{Var}(X_2) + 2\text{Cov}(X_1, X_2)\end{aligned}$$

- General: add variances + twice all covariance pairs.
- What if X_1, X_2 dependent, but uncorrelated?

Examples

- If $Y = X$ then:

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{Var}(X) + \mathrm{Var}(X) + 2\mathrm{Cov}(X,X) \\ &= 4\mathrm{Var}(X)\end{aligned}$$

- If $Y = -X$ then:

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{Var}(X) + \mathrm{Var}(-X) + 2\mathrm{Cov}(X,-X) \\ &= 0\end{aligned}$$

- $Y = X + W$, X and W are independent.

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,X+W) \\ &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,X) \\ &= 3\mathrm{Var}(X) + \mathrm{Var}(Y) \\ &= 4\mathrm{Var}(X) + \mathrm{Var}(W)\end{aligned}$$