LECTURE 7

- Readings: Finish Chapter 2

Lecture outline

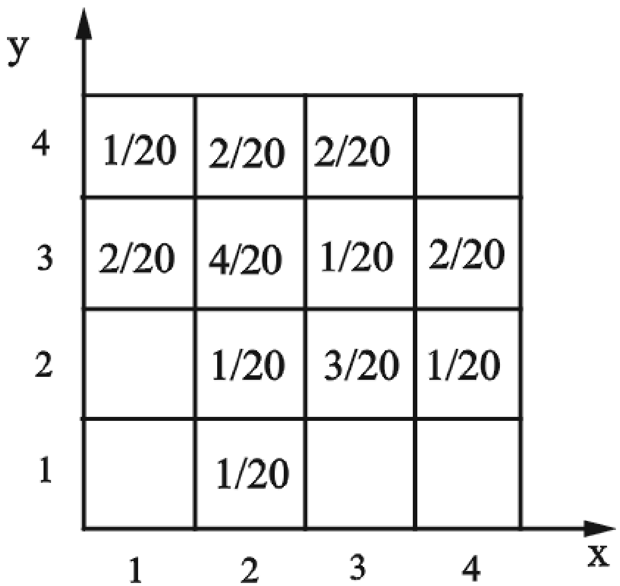
- Joint PMFs
- Independent random variables
- More expectations, variances
- Binomial distribution revisited
- The hat problem
- Application: Point-to-Point Communication

Review

- Random Variables and PMF
- Expectation
- Variance
- Examples:
  - Binomial, Geometric, and Poisson

Joint PMFs

- $p_{X,Y}(x,y) = \mathbf{P}(X = x \text{ and } Y = y)$



- $p_X(x) = \sum_y p_{X,Y}(x,y)$

- $p_{X|Y}(x|y)$
  $= \mathbf{P}(X = x | Y = y)$
  $= \frac{p_{X,Y}(x,y)}{p_Y(y)}$

- $\sum_x \sum_y p_{X,Y}(x,y) = 1$

- $\sum_x p_{X|Y}(x|y) = 1$

Independent Random Variables

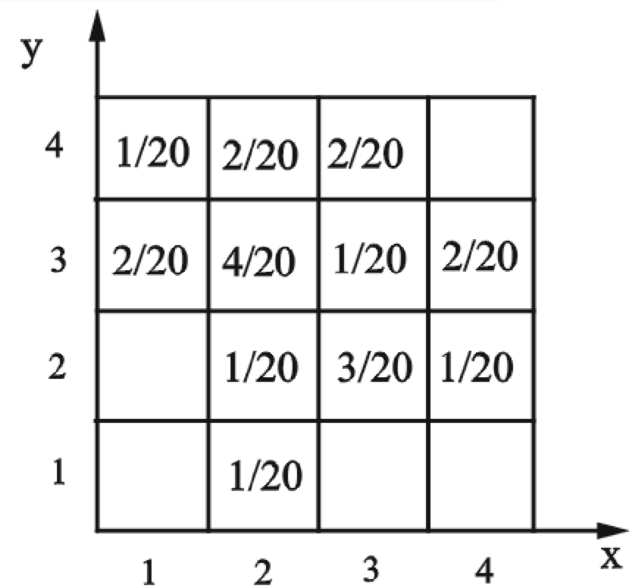
$$p_{X,Y,Z}(x,y,z) = p_X(x)p_{Y|X}(y|x)p_{Z|X,Y}(z|x,y)$$

- Random variables X, Y and Z are independent if (for all x, y and z):

$$p_{X,Y,Z}(x,y,z) = p_X(x) \cdot p_Y(y) \cdot p_Z(z)$$

- Example: Independent?

- What if we condition on $X \leq 2$ and $Y \geq 3$?



Expectations

$$\mathbf{E}[X] = \sum_x x \cdot p_X(x)$$

$$\mathbf{E}[g(X,Y)] = \sum_x \sum_y g(x,y) \cdot p_{X,Y}(x,y)$$

- In general: $\mathbf{E}[g(X,Y)] \neq g(\mathbf{E}[X], \mathbf{E}[Y])$
- $\mathbf{E}[\alpha X + \beta)] = \alpha \mathbf{E}[X] + \beta$
- $\mathbf{E}[X + Y + Z] = \mathbf{E}[X] + \mathbf{E}[Y] + \mathbf{E}[Z]$
- If X and Y are independent:
  - $\mathbf{E}[X \cdot Y] = \mathbf{E}[X] \cdot \mathbf{E}[Y]$
  - $\mathbf{E}[g(X) \cdot h(Y)] = \mathbf{E}[g(X)] \cdot \mathbf{E}[h(Y)]$

Variances

- $\text{var}(aX) = a^2\text{var}(X)$
- $\text{var}(X + a) = \text{var}(X)$
- Let $Z = X + Y$. If X and Y independent:

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$$

- Examples:
  - If $X = Y$, $\text{var}(X + Y) =$ $4\text{var}(X)$
  - If $X = -Y$, $\text{var}(X + Y) =$ 0
  - If X, Y indep., and $Z = X - 3Y$, $\text{var}(Z) =$ $\text{var}(X) + 9\text{var}(Y)$

Binomial Mean and Variance

- X = # of successes in n independent trials
  - Probability of success: p

$$\mathbf{E}[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

- $X_i = \begin{cases} 1, & \text{if success in trial } i, \\ 0, & \text{otherwise} \end{cases}$

- $\mathbf{E}[X_i] = p$
- $\text{var}(X_i) = p - p^2$
- $\mathbf{E}[X] = np$
- $\text{var}(X) = np(1-p)$

The Hat Problem

- n people throw their hats in a box and then pick one at random.
  - X: number of people who get their own hat
  - Find $\mathbf{E}[X]$

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat,} \\ 0, & \text{otherwise.} \end{cases}$$

- $X = X_1 + X_2 + \cdots + X_n$
- $\mathbf{P}(X_i = 1) = \frac{1}{n}$
- $\mathbf{E}[X_i] = \frac{1}{n}$
- Are the X_i independent? *No*
- $\mathbf{E}[X] = n(\frac{1}{n}) = 1$

Variance in the Hat Problem

- $\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \mathbf{E}[X^2] - 1$

$$X^2 = \sum_i X_i^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X_i X_j$$

- $\mathbf{E}[X_i^2] = \frac{1}{n}$

$$\mathbf{P}(X_1 X_2 = 1)$$
$$= \mathbf{P}(X_1 = 1) \cdot \mathbf{P}(X_2 = 1 | X_1 = 1) = (\tfrac{1}{n})(\tfrac{1}{n-1})$$

- $\mathbf{E}[X^2] = n\frac{1}{n} + n(n-1)(\frac{1}{n})(\frac{1}{n-1}) = 2$

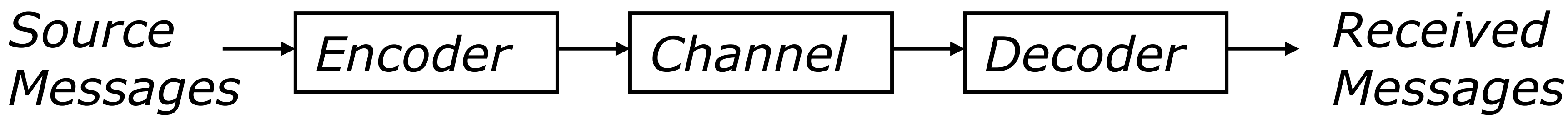
- $\text{var}(X) = 1$

Challenge: BBall Party

Your Guests are all BBall fans and they wear BBall Caps. There is a total of s teams in the league. Everyone of your guests is equally likely to be a fan of any one of these teams.

Compute the expected number of people who will pick a cap from their own team!

A Communication Example
Introduction



- A point-to-point communication system.
- Probabilistic model:
  - Messages are independent binary r.v.s.
  - The encoder is a deterministic function.
  - The channel introduces errors. It is modeled as a conditional pmf.
  - The decoder is a deterministic function.

A Communication Example

Messages, Encoding

- **Messages**: I.I.D. Bernoulli r.v.s $M_1, M_2, \cdots$

$$M_i = \begin{cases} 1, & \text{with probability p} \\ 0, & \text{with probability } 1 - \text{p} \end{cases}$$

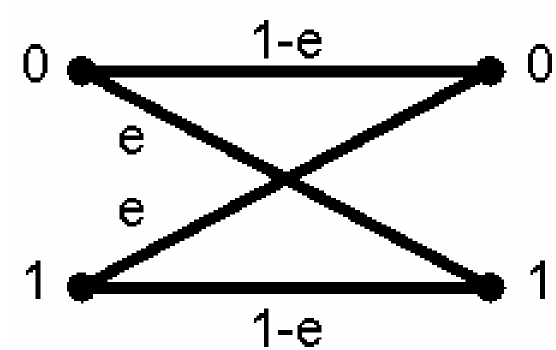
- **Encoding**:
  Repeat n times,

$$\{0,1\} \longmapsto \{0,1\}^n$$
$$0 \longrightarrow 00\cdots0$$
$$1 \longrightarrow 11\cdots1$$

A Communication Example

Channel - 1

- Encoded bits are transmitted independently one by one through the **channel**.
- The channel flips each bit independently, and with "crossover" probability e.
- Pictorially:



A Communication Example

Channel - 2

- Mathematically:

$$p_{Y|X}(y|x) = \begin{cases} 1 - e & \text{If } x = y. \\ e & \text{If } x \neq y. \end{cases}$$

- Multiple transmissions: $X_1, X_2, \cdots$

$$Y_1, Y_2, \cdots$$

$$\mathbf{P}_{Y_1, Y_2, \cdots | X_1, X_2, \cdots}(y_1, y_2, \cdots | x_1, x_2, \cdots)$$

$$= \mathbf{P}_{Y|X}(y_1|x_1) \cdot \mathbf{P}_{Y|X}(y_2|x_2) \cdot \cdots$$

A Communication Example

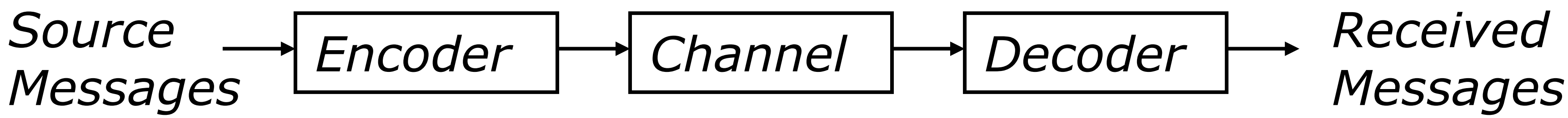
Decoding

- **Decoding**: Majority Rule
  - Consider a single message: M
  - Encoded r.v.s: $X_1, \ldots, X_n$
  - Received r.v.s: $Y_1, \ldots, Y_n$
  - Decoded message is a function of $Y_1, \ldots, Y_n$:

$$\widehat{M}_{Y_1,\cdots,Y_N}(y_1,\cdots,y_n) = \begin{cases} 0 & \text{If } y_1 + \cdots + y_n < n/2. \\ 1 & \text{If } y_1 + \cdots + y_n \geq n/2. \end{cases}$$

A Communication Example

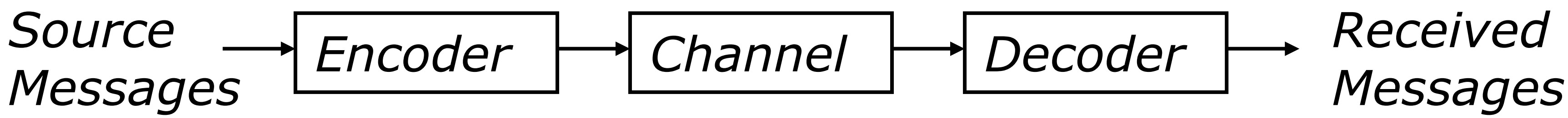
Performance



- If $n = 1$, what is $\mathrm{P}(\widehat{M} \neq M)$?
- What if $n = 3$?
- What if n is made arbitrarily large?
- Is there anything lost?
- How good is the decision rule?

A Communication Example

Probability of Error



- *Probability of error:* $\mathbf{P}(\hat{M} \neq M) = \mathbf{P}(\hat{M} = 1 | M = 0)(1 - p)$
$+\mathbf{P}(\hat{M} = 0 | M = 1)p$

$$\mathbf{P}(\hat{M} = 1 | M = 0) = \mathbf{P}(Y_1 + \cdots + Y_n > \tfrac{n}{2} | M = 0)$$

$$= \textstyle\sum_{k \geq \frac{n}{2}} \binom{n}{k} e^k (1 - e)^{n-k}$$