LECTURE 6

- Readings: Sections 2.4-2.6

Lecture outline

- Review PMF, expectation, variance
- Conditional PMF
- Geometric PMF
- Total expectation theorem
- Joint PMF of two random variables
- Independence

Review

- Random variable X: function from sample space to the real numbers
- PMF (for discrete random variables):

$$p_X(x) = \mathbf{P}(X = x), \ \sum_x \mathbf{P}(X = x) = 1$$

- Expectation: $\mathbf{E}[X] = \sum_x x \cdot p_X(x)$

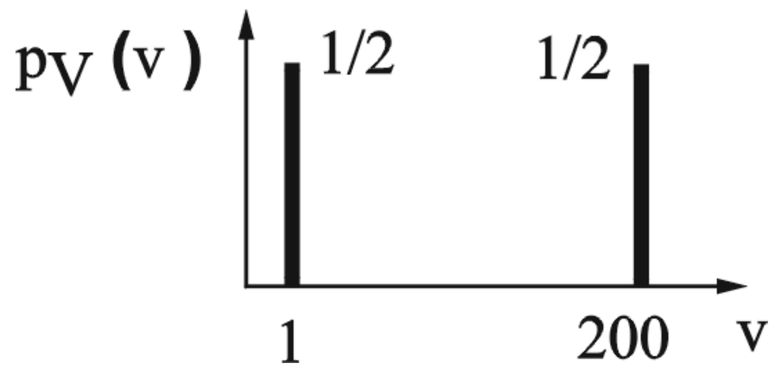
$$\mathbf{E}[g(X)] = \sum_x g(x) \cdot p_X(x)$$

- $\mathbf{E}[X - \mathbf{E}[X]] = 0$
- Variance:

$$\begin{aligned} \text{var}(X) &= \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] \\ &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 \\ &= \sum_x (x - \mathbf{E}[X])^2 \cdot p_X(x) \end{aligned}$$

Average Speed vs. Average Time - 1

- Traverse a 200 mile distance at constant but random speed V:



- $d = 200, T = t(V) = 200/V$
  - $\mathbf{E}[V] = 1 \cdot (1/2) + 200 \cdot (1/2) = 100.5$
  - $$\begin{aligned} \mathbf{E}[T] &= \mathbf{E}[t(V)] = \sum_v t(v) \cdot p_V(v) \\ &= \frac{200}{1} \cdot \frac{1}{2} + \frac{200}{200} \cdot \frac{1}{2} = 100.5 \end{aligned}$$

Average Speed vs. Average Time - 2

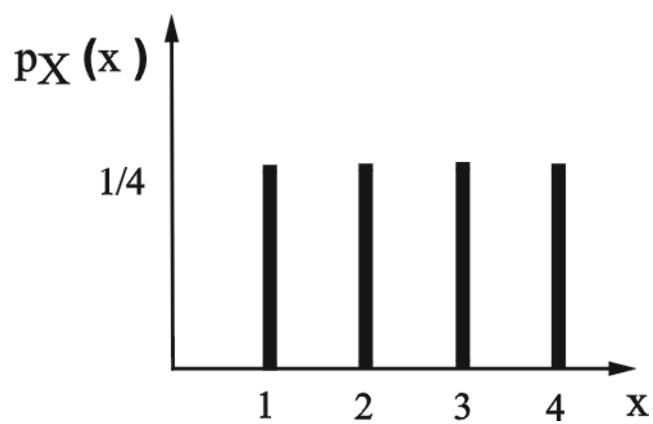
- $\mathbf{E}[T] \cdot \mathbf{E}[V] \neq 200 = d = \mathbf{E}[TV]$
- $\mathbf{E}[T] \neq 200/\mathbf{E}[V]$.

$$\begin{aligned} \text{var}(V) &= \textstyle\sum_v (v - \mathbf{E}[V])^2 \cdot p_V(v) \\ &= (1 - 100.5)^2 \frac{1}{2} + (200 - 100.5)^2 \frac{1}{2} \\ &\approx 10,000 \end{aligned}$$

- Standard Deviation $\sigma_V = \sqrt{\text{var}(V)} \approx 100$.

Conditional Expectation

- Recall: $p_{X|A}(x) = \mathbf{P}(X = x|A)$
- Definition: $\boxed{\mathbf{E}[X|A] = \sum_x x \cdot p_{X|A}(x)}$



$$\mathbf{E}[X|X \geq 2] =$$

Geometric PMF

- X: Waiting time for the #1 bus at the MIT stop

$$p_X(k) = (1-p)^{k-1}p, \qquad k = 1, 2, \ldots$$

- What is the expected waiting time, $\mathbf{E}[X]$?

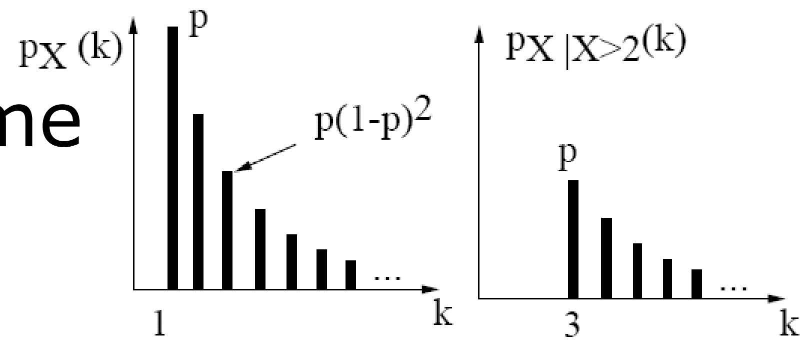
- What is the expected waiting time conditioned on the fact that you have already waited 2 minutes?

Geometric PMF

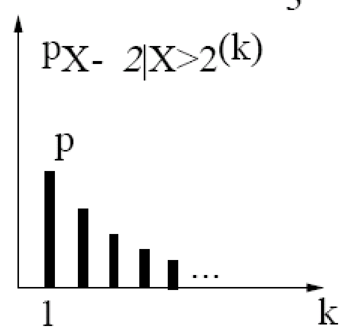
- Expected time:

$$\mathbf{E}[X] = \sum_{k=1}^{\infty} k \cdot p_X(k) = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} p$$

- Memoryless property:
  Given that X > 2,
  the r.v. X – 2 has same
  geometric PMF.

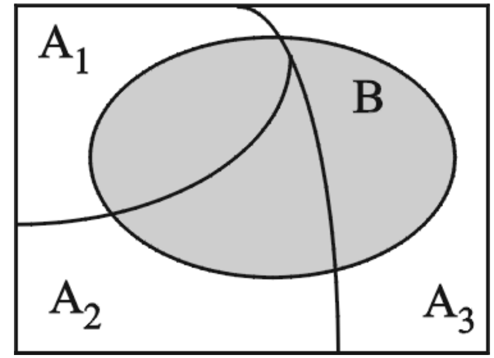


$$\mathbf{E}[X | X \geq 2] = 2 + \mathbf{E}[X]$$



Total Expectation Theorem

- Partition of sample space into disjoint events: $A_1, A_2, \cdots, A_n$



$$\mathbf{P}(B) = \mathbf{P}(A_1)\mathbf{P}(B|A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B|A_n)$$

$$\boxed{\mathbf{E}[X] = \mathbf{P}(A_1)\mathbf{E}[X|A_1] + \cdots + \mathbf{P}(A_n)\mathbf{E}[X|A_n]}$$

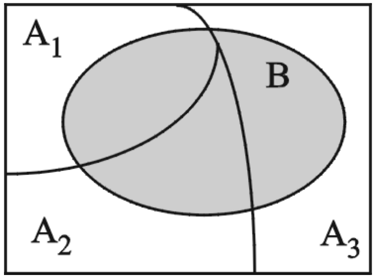
- Geometric example: $A_1 : \{X = 1\},\ A_2 : \{X > 1\}$

$$\begin{aligned}\mathbf{E}[X] &= \mathbf{P}(X = 1)\mathbf{E}[X|X = 1] \\ &\quad + \cdots + \mathbf{P}(X > 1)\mathbf{E}[X|X > 1]\end{aligned}$$

- Solve to get $\mathbf{E}[X] = 1/p$

Geometric R.V.

$$\mathbf{E}[X] = \mathbf{P}(A_1)\mathbf{E}[X|A_1] + \cdots + \mathbf{P}(A_n)\mathbf{E}[X|A_n]$$



- Geometric example:

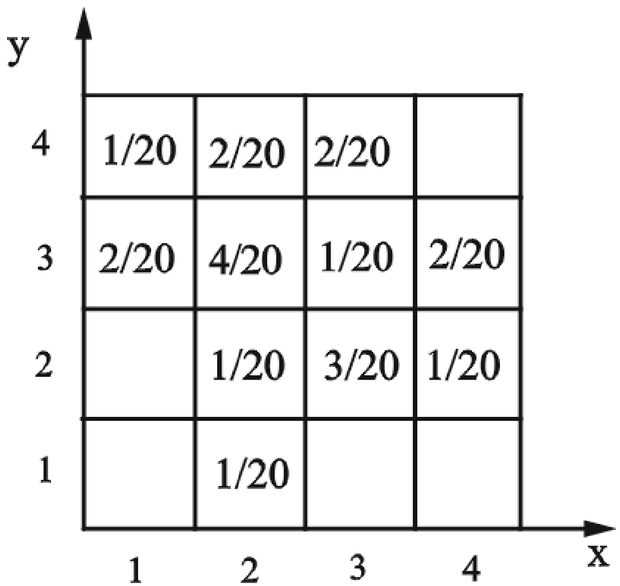
$$A_1 : \{X = 1\},\ A_2 : \{X > 1\}$$

$$\begin{aligned}\mathbf{E}[X] &= \mathbf{P}(X = 1)\mathbf{E}[X|X = 1] \\ &\quad + \cdots + \mathbf{P}(X > 1)\mathbf{E}[X|X > 1]\end{aligned}$$

- Solve to get $\mathbf{E}[X] = 1/p$

Joint PMFs

- $p_{X,Y}(x,y) = \mathbf{P}(X = x \text{ and } Y = y)$



- $p_X(x) = \sum_y p_{X,Y}(x,y)$

- $p_{X|Y}(x|y)$
  $= \mathbf{P}(X = x|Y = y)$
  $= \frac{p_{X,Y}(x,y)}{p_Y(x)}$

- $\sum_x \sum_y p_{X,Y}(x,y) = 1$

- $\sum_x p_{X|Y}(x|y) = 1$

Independent Random Variables

$$p_{X,Y,Z}(x,y,z) = p_X(x)p_{Y|X}(y|x)p_{Z|X,Y}(z|x,y)$$

- Random variables X, Y and Z are independent if (for all x, y and z):

$$\boxed{p_{X,Y,Z}(x,y,z) = p_X(x) \cdot p_Y(y) \cdot p_Z(z)}$$

- Example: Independent?

- What if we condition on $X \leq 2$ and $Y \geq 3$?