LECTURE 3

- Readings: Sections 1.5

Lecture outline

- Review
- Independence of two events
- Independence of a collection of events

Review

$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}$, assuming $\mathbf{P}(B) > 0$.

- Multiplication rule:

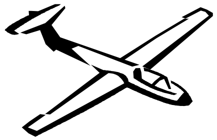
$$\mathbf{P}(A \cap B) = \mathbf{P}(B) \cdot \mathbf{P}(A \mid B) = \mathbf{P}(A) \cdot \mathbf{P}(B \mid A)$$

- Total probability theorem:

$$\mathbf{P}(B) = \mathbf{P}(A)\mathbf{P}(B \mid A) + \mathbf{P}(A^c)\mathbf{P}(B \mid A^c)$$

- Bayes rule:

$$\mathbf{P}(A_i \mid B) = \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\mathbf{P}(B)}$$



Extended Radar Example

- Threat alert affects the outcome

(0) (?) (1)

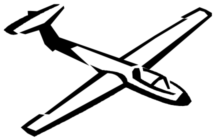
$\mathbf{P}(\cdots|\text{Threat})$

Airplane \ Radar	Low(0)	Medium(?)	High(1)
Absent	0.1125	0.05	0.0125
Present	0.055	0.22	0.55

$\mathbf{P}(\cdots|\text{No Threat})$

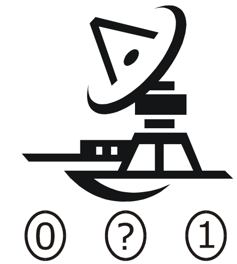
Airplane \ Radar	Low(0)	Medium(?)	High(1)
Absent	0.45	0.20	0.05
Present	0.02	0.08	0.20

- $\mathbf{P}(\text{Threat})$=Prior probability of threat= p



Extended Radar Example

(continued)



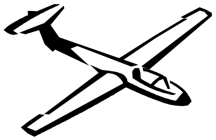
- A=Airplane, R=Radar Reading

$\mathbf{P}(A, R) = \mathbf{P}(\text{Threat})\mathbf{P}(A, R|\text{Threat}) + \mathbf{P}(\text{No Threat})\mathbf{P}(A, R|\text{No Threat})$

- If we let $p = \mathbf{P}(\text{Threat})$, then we get:

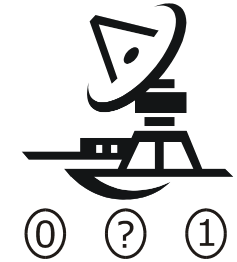
$\mathbf{P}(A, R)$

Radar / Airplane	Low(0)	Medium(?)	High(1)
Absent	0.45-0.3375p	0.20-0.15p	0.05-0.0375p
Present	0.02+0.0145p	0.08+0.14p	0.20+35p



Extended Radar Example

(continued)



$\mathbf{P}(A,R)$

Airplane \ Radar	Low(0)	Medium(?)	High(1)
Absent	0.45-0.3375p	0.20-0.15p	0.05-0.0375p
Present	0.02+0.0145p	0.08+0.14p	0.20+35p

- Given the Radar registered High, and a plane was absent, What is the probability that there was a threat?
- How does the decision region behave, as a function of p?

Independence of Two Events

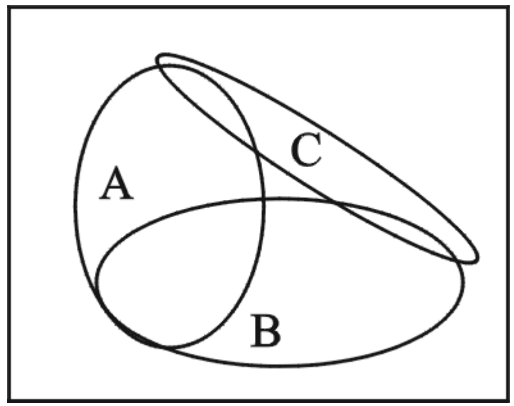
- Definition: $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B|A)$

- Recall:
  - Independence of *B* from *A*:
    $\mathbf{P}(B|A) = \mathbf{P}(B)$
  - By symmetry, $\mathbf{P}(A|B) = \mathbf{P}(A)$

- Examples:
  - *A* and *B* are disjoint.
  - Independence of A^c and *B*.
  - $\mathbf{P}(A|B) = \mathbf{P}(A|B^c)$

Conditioning may affect independence

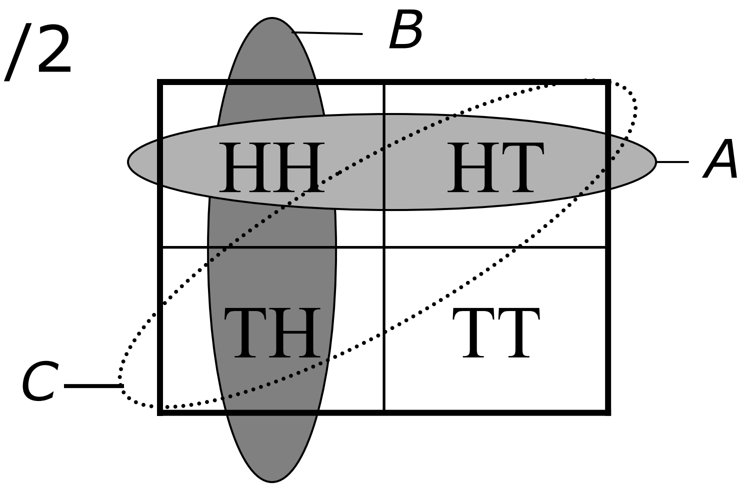
- Assume *A* and *B* are independent:



- If we are told that *C* occurred, are *A* and *B* independent?

Conditioning may affect independence

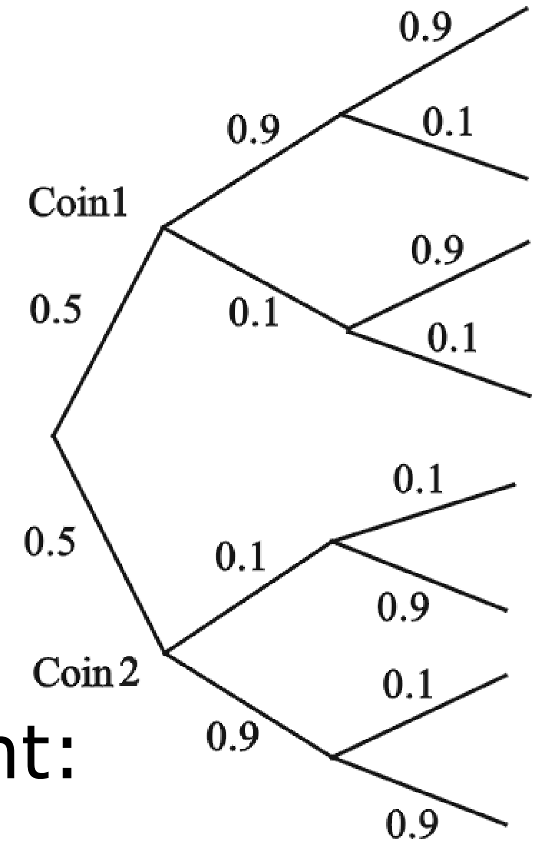
- Example 1:
  - Two independent fair ($p=½$) coin tosses.
  - Event A: First toss is H
  - Event B: Second toss is H
  - $\mathbf{P}(A) = \mathbf{P}(B) = 1/2$



  - Event C: The two outcomes are different.
  - Conditioned on C, are A and B independent?

Conditioning may affect independence

- Example 2:
  - Choice between two unfair coins, with equal probability.
  - $\mathbf{P}(H|\text{coin } 1) = 0.9$,
    $\mathbf{P}(H|\text{coin } 2) = 0.1$
  - Keep tossing the chosen coin.



- Are future tosses independent:
  - If we know we chose coin A?
  - If we do not know which coin we chose?
  - Compare: $\mathbf{P}(\text{toss } 11 = H)$
    $\mathbf{P}(\text{toss } 11 = H | \text{ first 10 tosses are H})$

Independence of a Collection of Events

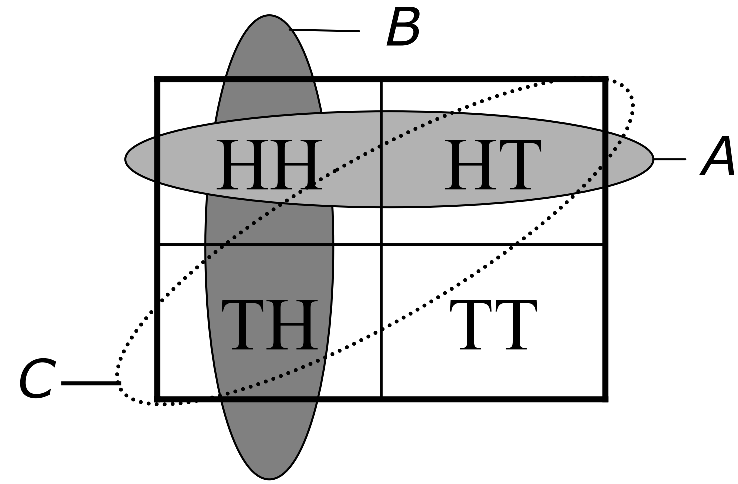
- Intuitive definition:
  - Information about some of the events tells us nothing about probabilities related to remaining events.
  - Example: $\mathbf{P}(A_1 \cap (A_2^c \cup A_3) \mid A_5 \cap A_6^c)$
    $= \mathbf{P}(A_1 \cap (A_2^c \cup A_3))$
- Mathematical definition:
  - For any distinct $i, j, \ldots, q$:

$$\mathbf{P}(A_i \cap A_j \cap \cdots \cap A_q) = \mathbf{P}(A_i)\mathbf{P}(A_j) \cdots \mathbf{P}(A_q)$$

Independence vs. Pairwise Independence

- Example 1 Revisited:
  - Two independent fair ($p=½$) coin tosses.
  - Event A: First toss is H
  - Event B: Second toss is H
  - Event C: The two outcomes are different.

- $\mathbf{P}(C) = \mathbf{P}(A) = \mathbf{P}(B) = \frac{1}{2}$
- $\mathbf{P}(C \cap A) = \frac{1}{4}$
- $\mathbf{P}(C \mid A \cap B) = 0$



- Pairwise independence **does not** imply independence.

The King’s Sibling

- The king comes from a family of two children.
- What is the probability that his sibling is female?