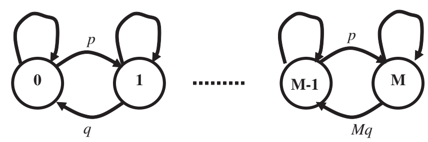
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Solutions for Problem Set 11:
Topic: Markov Processes
Due: May 12, 2006

1. (a) We define a Markov chain with states $0, 1, \cdots, m$, corresponding to the number of customers in the house. Assume that $mq < 1$, the transition probability graph is given as follows,



(b) For the above Markov Chain, the local balance equations are

$$\pi_i p = \pi_{i+1}(i+1)q, \quad i = 0, 1, \cdots, m-1.$$

We define $\rho = p/q$, and obtain $\pi_{i+1} = \frac{\rho}{i+1}\pi_i$, which leads to

$$\pi_i = \frac{\rho^i}{i!}\pi_0, \quad i = 0, 1, \cdots, m-1.$$

By using the normalization equation, $1 = \pi_0 + \pi_1 + \cdots + \pi_m$, we obtain

$$1 = \pi_0(1 + \frac{\rho^1}{1!} + \frac{\rho^2}{2!} + \cdots + \frac{\rho^m}{m!}),$$

and

$$\pi_0 = \frac{1}{\sum_{k=0}^{m} \frac{\rho^k}{k!}}.$$

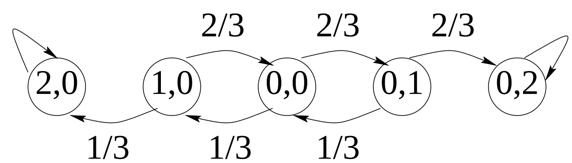
Using the equation $\pi_i = \frac{\rho^i}{i!}\pi_0$, the steady-state probabilities are

$$\pi_i = \frac{\frac{\rho^i}{i!}}{\sum_{k=0}^{m} \frac{\rho^k}{k!}}.$$

Therefore, the average number of customers in the house is given by

$$\bar{N} = \sum_{i=0}^{m} i\pi_i = \rho * \frac{\sum_{i=0}^{m-1} \frac{\rho^i}{i!}}{\sum_{k=0}^{m} \frac{\rho^k}{k!}}.$$

2. (a) Denote by (x, y) the score of Sam and Pat respectively, a Markov chain that describes the game is



Note that the game ends when either state $(0, 2)$ or $(2, 0)$ is entered.

(b) Since we have a finite number of states, a state is recurrent if and only if it is accessible from all the states that are accessible from it, and therefore, states $(0, 0), (0, 1)$ and $(1, 0)$ are transient and states $(0, 2)$ and $(2, 0)$ are recurrent.

(c) The probability that Pat wins is the probability that we get absorbed to state $(0, 2)$. Setting up the equations, we solve for $a_{(1,0)}, a_{(0,0)}$ and $a_{(0,1)}$

$$\begin{aligned} a_{(0,1)} &= \frac{2}{3} + \frac{1}{3}a_{(0,0)} \\ a_{(0,0)} &= \frac{2}{3}a_{(0,1)} + \frac{1}{3}a_{(1,0)} \\ a_{(1,0)} &= \frac{2}{3}a_{(0,0)} \end{aligned}$$

which yields that following

$$\begin{aligned} a_{(0,1)} &= \frac{14}{15} \\ a_{(0,0)} &= \frac{12}{15} \\ a_{(1,0)} &= \frac{8}{15} \end{aligned}$$

Therefore the probability of Pat winning is equal to $12/15 = 0.8$.