LECTURE 6

- **Readings:** Sections 2.4-2.6

Lecture outline

- Review: PMF, expectation, variance
- Conditional PMF
- Geometric PMF
- Total expectation theorem
- Joint PMF of two random variables

Review

- Random variable X: function from sample space to the real numbers
- PMF (for discrete random variables): $p_X(x) = \mathbf{P}(X = x)$
- Expectation:

$$\mathbf{E}[X] = \sum_x x p_X(x)$$

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

$$\mathbf{E}[\alpha X + \beta] = \alpha \mathbf{E}[X] + \beta$$

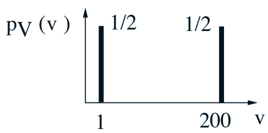
- $\mathbf{E}\Big[X - \mathbf{E}[X]\Big] =$

$$\begin{aligned} \text{var}(X) &= \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] \\ &= \sum_x (x - \mathbf{E}[X])^2 p_X(x) \\ &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 \end{aligned}$$

Standard deviation: $\sigma_X = \sqrt{\text{var}(X)}$

Random speed

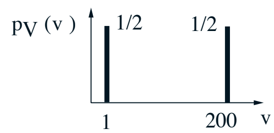
- Traverse a 200 mile distance at constant but random speed V



- $d = 200$, $T = t(V) = 200/V$
- $\mathbf{E}[V] =$
- $\text{var}(V) =$
- $\sigma_V =$

Average speed vs. average time

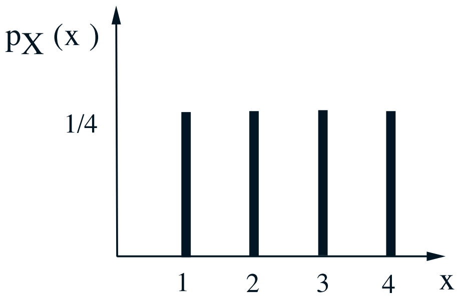
- Traverse a 200 mile distance at constant but random speed V



- time in hours $= T = t(V) =$
- $\mathbf{E}[T] = \mathbf{E}[t(V)] = \sum_v t(v) p_V(v) =$
- $\mathbf{E}[TV] = 200 \neq \mathbf{E}[T] \cdot \mathbf{E}[V]$
- $\mathbf{E}[200/V] = \mathbf{E}[T] \neq 200/\mathbf{E}[V]$.

Conditional PMF and expectation

- $p_{X|A}(x) = \mathbf{P}(X = x \mid A)$
- $\mathbf{E}[X \mid A] = \sum_x x p_{X|A}(x)$



- Let $A = \{X \geq 2\}$

$p_{X|A}(x) =$

$\mathbf{E}[X \mid A] =$

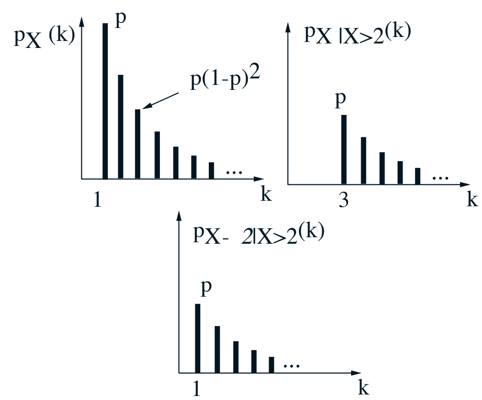
Geometric PMF

- X: number of independent coin tosses until first head

$$p_X(k) = (1-p)^{k-1}p, \qquad k = 1, 2, \ldots$$

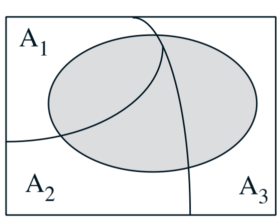
$$\mathbf{E}[X] = \sum_{k=1}^{\infty} k p_X(k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$$

- Memoryless property: Given that $X > 2$, the r.v. $X - 2$ has same geometric PMF



Total Expectation theorem

- Partition of sample space into disjoint events $A_1, A_2, \ldots, A_n$



$\mathbf{P}(B) = \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n)$

$p_X(x) = \mathbf{P}(A_1)p_{X|A_1}(x) + \cdots + \mathbf{P}(A_n)p_{X|A_n}(x)$

$\mathbf{E}[X] = \mathbf{P}(A_1)\mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\mathbf{E}[X \mid A_n]$

- Geometric example:
  $A_1: \{X = 1\}, \quad A_2: \{X > 1\}$

$$\begin{aligned} \mathbf{E}[X] &= \mathbf{P}(X = 1)\mathbf{E}[X \mid X = 1] \\ &\quad + \mathbf{P}(X > 1)\mathbf{E}[X \mid X > 1] \end{aligned}$$

- Solve to get $\mathbf{E}[X] = 1/p$

Joint PMFs

- $p_{X,Y}(x, y) = \mathbf{P}(X = x \text{ and } Y = y)$

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

- $\sum_x \sum_y p_{X,Y}(x, y) =$
- $p_X(x) = \sum_y p_{X,Y}(x, y)$
- $p_{X|Y}(x \mid y) = \mathbf{P}(X = x \mid Y = y) = \dfrac{p_{X,Y}(x, y)}{p_Y(y)}$
- $\sum_x p_{X|Y}(x \mid y) =$