LECTURE 25
Outline

- **Reference:** Section 9.4
- Course Evaluations (until 12/16)
  http://web.mit.edu/subjectevaluation
- Review of simple binary hypothesis tests
  - examples
- Testing composite hypotheses
  - is my coin fair?
  - is my die fair?
  - goodness of fit tests

Simple binary hypothesis testing

- **null hypothesis** H_0:
  $X \sim p_X(x; H_0)$ [or $f_X(x; H_0)$]
- **alternative hypothesis** H_1:
  $X \sim p_X(x; H_1)$ [or $f_X(x; H_1)$]
- Choose a **rejection region** R;
  reject H_0 iff data $\in R$

- Likelihood ratio test: reject H_0 if

$$\frac{p_X(x; H_1)}{p_X(x; H_0)} > \xi \quad \text{or} \quad \frac{f_X(x; H_1)}{f_X(x; H_0)} > \xi$$

  - fix false rejection probability α
    (e.g., $\alpha = 0.05$)
  - choose ξ so that $\mathbf{P}(\text{reject } H_0; H_0) = \alpha$

Example (test on normal mean)

- n data points, i.i.d.
  H_0: $X_i \sim N(0,1)$
  H_1: $X_i \sim N(1,1)$
- Likelihood ratio test; rejection region:

$$\frac{(1/\sqrt{2\pi})^n \exp\{-\sum_i (X_i - 1)^2/2\}}{(1/\sqrt{2\pi})^n \exp\{-\sum_i X_i^2/2\}} > \xi$$

  - algebra: reject H_0 if: $\sum_i X_i > \xi'$
- Find ξ' such that

$$\mathbf{P}\Big(\sum_{i=1}^n X_i > \xi'; H_0\Big) = \alpha$$

  - use normal tables

Example (test on normal variance)

- n data points, i.i.d.
  H_0: $X_i \sim N(0,1)$
  H_1: $X_i \sim N(0,4)$
- Likelihood ratio test; rejection region:

$$\frac{(1/2\sqrt{2\pi})^n \exp\{-\sum_i X_i^2/(2 \cdot 4)\}}{(1/\sqrt{2\pi})^n \exp\{-\sum_i X_i^2/2\}} > \xi$$

  - algebra: reject H_0 if $\sum_i X_i^2 > \xi'$
- Find ξ' such that

$$\mathbf{P}\Big(\sum_{i=1}^n X_i^2 > \xi'; H_0\Big) = \alpha$$

  - the distribution of $\sum_i X_i^2$ is known
    (derived distribution problem)
  - "chi-square" distribution;
    tables are available

Composite hypotheses

- Got $S = 472$ heads in $n = 1000$ tosses; is the coin fair?
  - $H_0 : p = 1/2$ versus $H_1 : p \neq 1/2$
- Pick a **"statistic"** (e.g., S)
- Pick shape of **rejection region** (e.g., $|S - n/2| > \xi$)
- Choose **significance level** (e.g., $\alpha = 0.05$)
- Pick **critical value** ξ so that:

$$\mathbf{P}(\text{reject } H_0; H_0) = \alpha$$

  Using the CLT:

$$\mathbf{P}(|S - 500| \leq 31; H_0) \approx 0.95; \quad \xi = 31$$

- In our example: $|S - 500| = 28 < \xi$
  H_0 **not rejected** (at the 5% level)

Is my die fair?

- Hypothesis H_0:
  $\mathbf{P}(X = i) = p_i = 1/6,\ i = 1, \ldots, 6$
- Observed occurrences of i: N_i
- Choose form of rejection region; chi-square test:

$$\text{reject } H_0 \text{ if } T = \sum_i \frac{(N_i - np_i)^2}{np_i} > \xi$$

- Choose ξ so that:

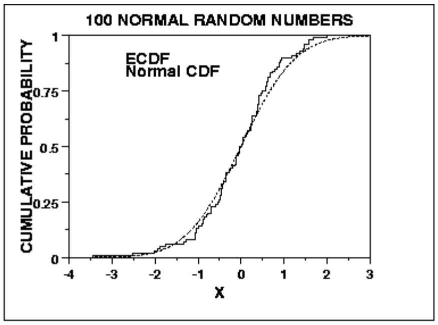
$$\mathbf{P}(\text{reject } H_0; H_0) = 0.05$$

$$\mathbf{P}(T > \xi; H_0) = 0.05$$

- Need the distribution of T:
  (CLT + derived distribution problem)
  - for large n, T has approximately a chi-square distribution
  - available in tables

Do I have the correct pdf?

- Partition the range into bins
  - np_i: expected incidence of bin i (from the pdf)
  - N_i: observed incidence of bin i
  - Use chi-square test (as in die problem)
- Kolmogorov-Smirnov test:
  form **empirical CDF**, $\hat{F}_X$, from data



(http://www.itl.nist.gov/div898/handbook/)

- $D_n = \max_x |F_X(x) - \hat{F}_X(x)|$
- $\mathbf{P}(\sqrt{n} D_n \geq 1.36) \approx 0.05$

What else is there?

- Systematic methods for coming up with shape of rejection regions
- Methods to estimate an unknown PDF (e.g., form a histogram and "smooth" it out)
- Efficient and recursive signal processing
- Methods to select between less or more complex models
  - (e.g., identify relevant "explanatory variables" in regression models)
- Methods tailored to high-dimensional unknown parameter vectors and huge number of data points (data mining)
- etc. etc....

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