

Tutorial 2 Solutions
September 23/24, 2010

1. A player is randomly dealt 13 cards from a standard 52-card deck.

(a) What is the probability the 13th card dealt is a king?

Answer: $\frac{4}{52}$.

Solution: Since we are not told anything about the first 12 cards that are dealt, the probability that the 13th card dealt is a King, is the same as the probability that the first card dealt, or in fact any particular card dealt is a King, and this equals: $\frac{4}{52}$.

(b) What is the probability the 13th card dealt is the first king dealt?

Answer: $\frac{1}{13} \cdot 4 \binom{48}{12} / \binom{52}{13}$.

Solution: The probability that the 13th card is the first king to be dealt is the probability that out of the first 13 cards to be dealt, exactly one was a king, and that the king was dealt last. Now, given that exactly one king was dealt in the first 13 cards, the probability that the king was dealt last is just $1/13$, since each “position” is equally likely. Thus, it remains to calculate the probability that there was exactly one king in the first 13 cards dealt. To calculate this probability we count the “favorable” outcomes and divide by the total number of possible outcomes. We first count the favorable outcomes, namely those with exactly one king in the first 13 cards dealt. We can choose a particular king in 4 ways, and we can choose the other 12 cards in $\binom{48}{12}$ ways, therefore there are $4 \cdot \binom{48}{12}$ favorable outcomes. There are $\binom{52}{13}$ total outcomes, so the desired probability is

$$\frac{1}{13} \cdot \frac{4 \binom{48}{12}}{\binom{52}{13}}.$$

For an alternative solution, we argue as in Example 1.10. The probability that the first card is not a king is $48/52$. Given that, the probability that the second is not a king is $47/51$. We continue similarly until the 12th card. The probability that the 12th card is not a king, given that none of the preceding 11 was a king, is $37/41$. (There are $52 - 11 = 41$ cards left, and $48 - 11 = 37$ of them are not kings.) Finally, the conditional probability that the 13th card is a king is $4/40$. The desired probability is

$$\frac{48 \cdot 47 \cdots 37 \cdot 4}{52 \cdot 51 \cdots 41 \cdot 40}.$$

2. Consider a random variable X such that

$$p_X(x) = \frac{x^2}{a} \text{ for } x \in \{-3, -2, -1, 1, 2, 3\}, \quad \mathbf{P}(X = x) = 0 \text{ for } x \notin \{-3, -2, -1, 1, 2, 3\},$$

where $a > 0$ is a real parameter.

(a) Find a .

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Solution. The sum of the values of the PMF of a random variable over all values that it takes with positive probability must be equal to 1. Hence, we have

$$\begin{aligned} 1 &= \sum_{x=-3}^3 p_X(x) \\ &= \frac{9}{a} + \frac{4}{a} + \frac{1}{a} + \frac{1}{a} + \frac{4}{a} + \frac{9}{a} \\ &= \frac{28}{a}, \end{aligned}$$

which implies that $a = 28$.

(b) What is the PMF of the random variable $Z = X^2$?

Solution. The following table shows the value of Z for a given value of X and the probability of that event.

x	-3	-2	-1	1	2	3
$p_X(x)$	9/28	1/7	1/28	1/28	1/7	9/28
$Z X=x$	9	4	1	1	4	9

We see that Z can take only three possible values with non-zero probability, namely 1, 4, and 9. In addition, for each value, there correspond two values of X . So we have, for example, $p_Z(9) = \mathbf{P}(Z = 9) = \mathbf{P}(X = -3) + \mathbf{P}(X = 3) = p_X(-3) + p_X(3)$. Hence the PMF of Z is given by

$$p_Z(z) = \begin{cases} 1/14 & \text{if } z = 1, \\ 2/7 & \text{if } z = 4, \\ 9/14 & \text{if } z = 9. \end{cases}$$

3. Suppose we label the classes A , B , and C . Now the probability that Joe and Jane will both be in class A is the number of possible combinations for class A that involve both Joe and Jane, divided by the total number of combinations for class A . Therefore the probability we are after is:

$$\frac{\binom{88}{28}}{\binom{90}{30}}.$$

Since there are three classrooms, the probability that Joe and Jane end up in the same classroom is simply three times the answer we found above:

$$3 \cdot \frac{\binom{88}{28}}{\binom{90}{30}}.$$

Another way of looking at the problem is described as follows,

Assume one of them pick first, say Joe. He can pick any one of the 90 available places. Then it's Jane's turn to pick. She has a probability of $\frac{29}{89}$ of picking in the same class as Joe. Therefore, the overall probability is $\frac{29}{89}$, which is the same as $3 \cdot \frac{\binom{88}{28}}{\binom{90}{30}}$.

4. Let A = event the 7 cards include exactly 3 aces.

$$P(A) = \frac{(\# \text{ ways to choose 3 aces}) \cdot (\# \text{ ways to choose other 4 cards})}{\# \text{ ways to choose 7 cards}} = \frac{\binom{4}{3} \binom{48}{4}}{\binom{52}{7}}.$$

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