Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

Recitation 16 Solutions
(6.041/6.431 Spring 2007 Quiz 2 Solutions)
November 2, 2010

Problem 1:

(a) (i) The plot for the PDF of X is shown in Figure 1. The PDF has to integrate to 1, so the area under $f_X(x)$ is 2c+c, which must equal 1. Therefore c = 1/3.
Integration of the PDF:

$$\begin{aligned} \int_2^4 f_X(x)dx &= 1 \\ \text{which breaks up to } \int_2^3 2cdx + \int_3^4 cdx &= 1 \\ = 2c + c &= 1 \\ \text{and } c &= 1/3. \end{aligned}$$

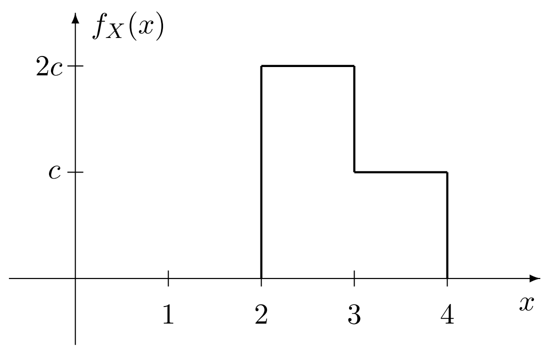

Figure 1: PDF of X

(ii)

$$\begin{aligned} \mathbf{E}[X] &= \int_2^4 x f_X(x)\ dx = \int_2^3 x \cdot 2/3\ dx + \int_3^4 x \cdot 1/3\ dx \\ &= 1/3 \cdot (3^2 - 2^2) + 1/6 \cdot (4^2 - 3^2) = 5/3 + 17/6 \\ &= 17/6. \end{aligned}$$

(iii)

$$\begin{aligned} \mathbf{E}[X^2] &= \int_2^4 x^2 f_X(x)\ dx = \int_2^3 x^2 \cdot 2/3\ dx + \int_3^4 x^2 \cdot 1/3\ dx \\ &= 2/9 \cdot (3^3 - 2^3) + 1/9 \cdot (4^3 - 3^3) = 38/9 + 37/9 \\ &= 25/3. \end{aligned}$$

(iv) Let $Y = 2X + 1$. The range of Y is not from 2 to 4, but now $5 \le y \le 9$. The shape of the PDF of Y should look like the PDF of X, but scaled by a factor such that it normalizes to 1. The range of Y is double the range of X, so the density is half. Plot shown below in Figure 2.

Since $Y = g(X)$ is a linear function of X, we can use the formula for the derived distribution for a linear function. $Y = 2X + 1$, so $f_Y(y) = \frac{1}{2} f_X(\frac{y-1}{2})$ for $5 \le y \le 9$. Figure 2 matches this distribution.

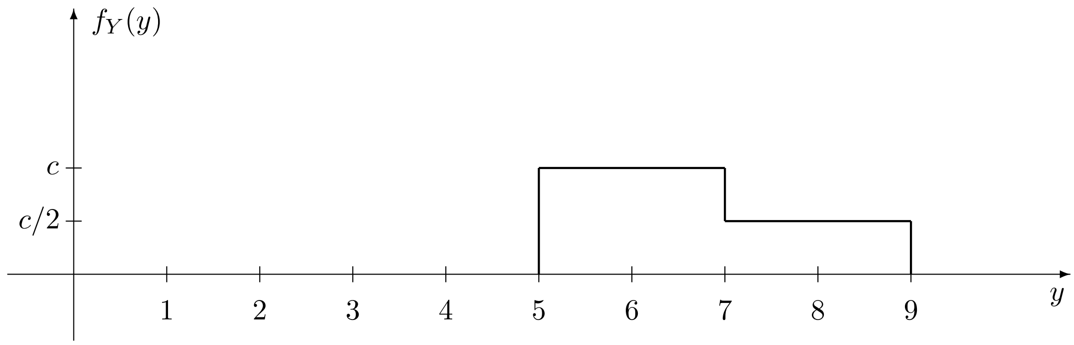

Figure 2: PDF of $Y = 2X + 1$

(b) First we calculate the joint PDF. It should have a non-zero joint density for the region, $2 \le x \le 4$ and $2 \le w \le 4$. However, it is not uniform within this entire square, as we have seen often in class. Due to the piece-wise uniform density of X, the square is partitioned into two rectangles of uniform joint densities. X and W are independent, so the joint density is just the product of the marginals.

$$
\begin{aligned}
f_{X,W}(x, w) &= f_X(x) f_W(w) \\
&= f_X(x) \cdot 1/2 \\
&= \begin{cases} c1 = 2/3 \cdot 1/2 = 1/3 &, \quad 2 \le x \le 3, 2 \le w \le 4. \\ c2 = 1/3 \cdot 1/2 = 1/6 &, \quad 3 \le x \le 4, 2 \le w \le 4. \end{cases}
\end{aligned}
$$

Variables $c1$ and $c2$ are used to denote the different joint densities, and are shown in the joint plot.
As a check, the joint PDF should be normalized to 1, which it is.
The joint PDF for X and W is shown in Figure 3.

Looking at the plot of the joint PDF, $\mathbf{P}(X \le W)$ is the region above the $X = W$ line. See Figure 4. We calculate the probability of interest by weighting the areas of the two parts of the shaded regions by c_1 and c_2:

$$
\begin{aligned}
\mathbf{P}(X \le W) &= 1/2 \cdot 1/6 + 3/2 \cdot 1/3 = 1/12 + 1/2 \\
&= 7/12.
\end{aligned}
$$

The graphical way is the easy solution. Of course, one can integrate:

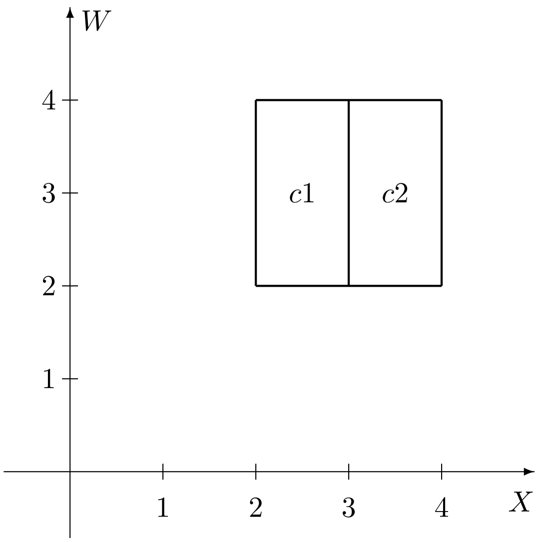

Figure 3: Joint PDF of X and W

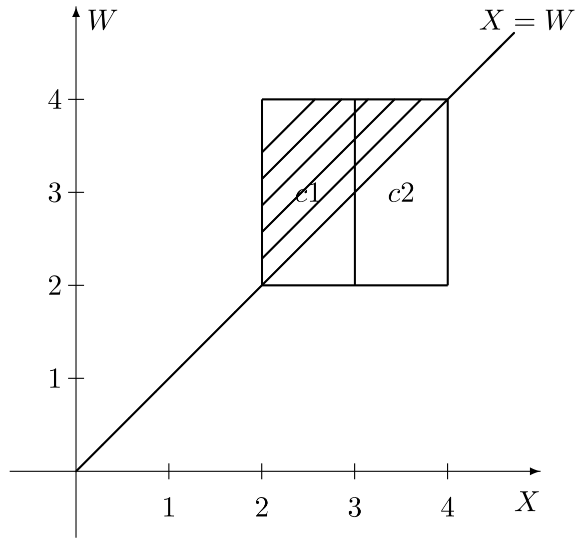

Figure 4: $\mathbf{P}(X \le W)$

$$\begin{aligned}\mathbf{P}(X \le W) &= \int_2^3 \int_x^4 1/3\ dwdx + \int_3^4 \int_x^4 1/6\ dwdx \\ &= \frac{1}{3}\int_2^3 (4-x)\ dx + \frac{1}{6}\int_3^4 (4-x)\ dx \\ &= 7/12\end{aligned}$$

(c) Be careful here, that T is the race time measured by the stopwatch, not just the over-estimated race time. Remember also that T and W are independent.

$$\begin{aligned} f_{W|T}(w|3) &= \frac{f_{W,T}(w,3)}{f_T(3)} \\ \text{where } f_{W,T}(w,3) &= f_W(w)f_T(3) = 10 \cdot 1/2 = 5 \text{ for } 2 \le w \le 4. \\ \text{and where } f_T(3) &= \int_{3-1/10}^{3} f_{W,T}(w,3)\ dw = 5 \cdot (1/10) = 1/2. \end{aligned}$$

Therefore,

$$f_{W|T}(w|3) = \begin{cases} 10, & \text{if } (3 - 1/10) \le w \le 3 \ \text{ and } t = 3, \\ 0, & \text{otherwise.} \end{cases}$$

(d) N is Normal(1/60, 4/3600). We standardize N to have mean 1 and standard deviation 1 to utilize the Normal table.

$$\begin{aligned}\mathbf{P}(N > \frac{5}{60}) &= 1 - \mathbf{P}(N < \frac{5}{60}) \\ &= 1 - \mathbf{P}(\frac{N - 1/60}{2/60} < \frac{5/60 - 1/60}{2/60}) \\ &= 1 - \Phi(2).\end{aligned}$$

Looking it up, $\Phi(2) = 0.9772$.

$$\text{So, } \mathbf{P}(N > \frac{5}{60}) = 1 - 0.9772 = 0.0028.$$

(e) Use derived distributions to find the CDF of S, then differentiate with respect to s to find the PDF of S. The range of S is determined from the range of W. Since $2 \leq w \leq 4$ for a nonzero PDF of W, $24/4 \leq s \leq 24/2$ for a nonzero PDF of S.

$$\begin{aligned}\mathbf{P}(S \leq s) &= \mathbf{P}(24/W \leq s) = \mathbf{P}(W \geq 24/s) \\ &= 1 - F_W(24/s) = 1 - \int_2^{24/s} f_W(w)dw \\ &= 1 - (12/s - 1) = 2 - 12/s\end{aligned}$$

Taking the derivative with respect to s,

$$\begin{aligned}f_S(s) &= \frac{d}{ds}(2 - 12/s) \\ &= \begin{cases} 12/s^2, & \text{if } 6 \leq s \leq 12 \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Problem 2.

(a) (i) This is a random sums problem so the mean and variance of A is found using the laws of iterated expectations and total variance.

$$\begin{aligned}
\mu_a &= \mathbf{E}[A] = \mathbf{E}[\mathbf{E}[A \mid N]] = \mathbf{E}[N\mathbf{E}[A_i]] = \mathbf{E}[A_i]\mathbf{E}[N] \\
&= 1/p. \\
\sigma_a^2 &= \text{var}(A) = \mathbf{E}[\text{var}(A \mid N)] + \text{var}(\mathbf{E}[A \mid N]) = \mathbf{E}[N\ \text{var}(A_i)] + \text{var}(N\mathbf{E}[A_i]) \\
&= \text{var}(A_i)\mathbf{E}[N] + \mathbf{E}[A_i]^2\text{var}(N) = 1/p + p/(1-p) \\
&= 1/p^2.
\end{aligned}$$

(ii)

$$\begin{aligned}
c_{ab} = \mathbf{E}[AB] &= \mathbf{E}[(A_1 + A_2 + A_3 + ...A_N)(B_1 + B_2 + B_3 + ...B_N)] \\
&= \mathbf{E}[\mathbf{E}[(A_1 + A_2 + A_3 + ...A_N)(B_1 + B_2 + B_3 + ...B_N)|N]] \\
&= \mathbf{E}[N\mathbf{E}[A_i]N\mathbf{E}[B_i]] = \mathbf{E}[N^2\mathbf{E}[A_i]\mathbf{E}[B_i]] = \mathbf{E}[A_i]\mathbf{E}[B_i]\mathbf{E}[N^2] \\
&= 1 \cdot 1 \cdot (\text{var}(N) + \mathbf{E}[N]^2) = (1-p)/p^2 + 1/p^2 \\
&= (2-p)/p^2.
\end{aligned}$$

(b) (i) If $N = 1$, $A = A_1$, which has a Normal distribution with mean 1 and variance 1.
If $N = 2$, $A = A_1 + A_2$, which is the sum of two Normals. Therefore the distribution of A is Normal($1+1$,$1+1$) or Normal(2,2).
Using total probability theorem, we find:

$$\begin{aligned}
f_A(a) &= f_{A|N=1}(a)P_N(1) + f_{A|N=2}(a)P_N(2) \\
&= \text{Normal(1,1)} \cdot 1/3 + \text{Normal(2,2)} \cdot 2/3 \\
&= \frac{1}{3\sqrt{2\pi}}e^{-(a-1)^2/2} + \frac{2}{3\sqrt{4\pi}}e^{-(a-2)^2/4}.
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathbf{P}(N = 1 \mid A = a) &= \frac{\mathbf{P}(A = a, N = 1)\delta}{\mathbf{P}(A = a)\delta}. \\
\text{where } \mathbf{P}(A = a)\delta &= f_A(a) \text{ was found in part (a)} \\
\text{and the joint is } P(A = a)P(N = 1)\delta &= f_A(a)P_N(1). \\
\text{Then, } \mathbf{P}(N = 1 \mid A = a) &= \frac{\frac{1}{3\sqrt{2\pi}}e^{-(a-1)^2/2}}{\frac{1}{3\sqrt{2\pi}}e^{-(a-1)^2/2} + \frac{2}{3\sqrt{4\pi}}e^{-(a-2)^2/4}}.
\end{aligned}$$

(c) Yes they are equal.
As a first check, they are both random variables. A and B are not independent from one another because they both depend on the RV N for the random sum. But, if we condition on N, then A and B are independent (hence they are conditionally independent). Is that what the right side of the equation states?
These expectations are equal if the PDFs of $A \mid N$ and $A \mid (B, N)$ are equal. Once N is known,

knowing B doesn't change what ones knows about A, so this not only shows that A and B are conditionally independent, given N, but $A \mid N$ has the same information as $A, B \mid N$.

Conditional independence of events X and Y on Z is defined as:

$$\begin{aligned} \mathbf{P}(X \cap Y \mid Z) = \mathbf{P}(X \mid Z)\mathbf{P}(Y \mid Z) \\ \text{or, equivalently} \\ \mathbf{P}(X \mid Y \cap Z) = \mathbf{P}(X \mid Z) \end{aligned}$$

Therefore, we show that the equality holds here.

$$\begin{aligned} \mathbf{E}[A \mid N] &= \mathbf{E}[A \mid B, N] \\ \int a f_{A|N}(a \mid n)\, da &= \int a f_{A|B,N}(a|b,n)\, da \end{aligned}$$

The above statement is equal if the PDFs are equal:

$$\begin{aligned} f_{A|N}(a \mid n) &= f_{A|B,N}(a \mid b,n) = \frac{f_{A,B,N}(a,b,n)}{f_{B,N}(b,n)} \\ &= \frac{f_{A,B|N}(a,b \mid n)P_N(n)}{f_{B|N}(b \mid n)P_N(n)} = \frac{f_{A|N}(a \mid n)f_{B|N}(b \mid n)}{f_{B|N}(b \mid n)} \\ &= f_{A|N}(a \mid n). \end{aligned}$$

So $\mathbf{E}[A \mid N] = \mathbf{E}[A \mid B, N]$ is true.