**Recitation 15 Solutions**
**October 28, 2010**

1. (a) Let $X$ be the time until the first bulb failure. Let $A$ (respectively, $B$) be the event that the frist bulb is of type $A$ (respectively, $B$). Since the two bulb types are equally likely, the total expectation theorem yields

$$\mathbf{E}[X] = \mathbf{E}[X|A]\mathbf{P}(A) + \mathbf{E}[X|B]\mathbf{P}(B) = 1 \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{2}{3}.$$

(b) Let $D$ be the event of no bulb failures before time $t$. Using the total probability theorem, and the exponential distributions for bulbs of the two types, we obtain

$$\mathbf{P}(D) = \mathbf{P}(D|A)\mathbf{P}(A) + \mathbf{P}(D|B)\mathbf{P}(B) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

(c) We have

$$\mathbf{P}(A|D) = \frac{\mathbf{P}(A \cap D)}{\mathbf{P}(D)} = \frac{\frac{1}{2}e^{-t}}{\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}} = \frac{1}{1+e^{-2t}}.$$

(d) The lifetime of the first type-A bulb is $X_A$, with PDF given by:

$$f_{X_A}(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Let $Y$ be the total lifetime of two type-B bulbs. Because the lifetime of each type-B bulb is exponential with $\lambda = 3$, the sum $Y$ has an Erlang distribution of order 2 with $\lambda = 3$. Its PDF is:

$$f_Y(y) = \begin{cases} 9ye^{-3y} & y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned}
P(G) &= P(Y \geq X_A) \\
&= \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{y} f_{X_A}(x)\,dx\,dy \\
&= \int_{0}^{\infty} 9ye^{-3y} \int_{0}^{y} e^{-x}\,dx\,dy = 9\int_{0}^{\infty} ye^{-3y} - e^{-x}\Big|_{x=0}^{x=y} dy \\
&= 9\int_{0}^{\infty} ye^{-3y}(1-e^{-y})\,dy = 9\int_{0}^{\infty} ye^{-3y} - ye^{-4y}\,dy \\
&= 9\left(-\frac{1}{3}ye^{-3y} - \frac{1}{9}e^{-3y} + \frac{1}{4}ye^{-4y} + \frac{1}{16}e^{-4y}\right)\Big|_{y=0}^{y=\infty} \\
&= 9\left(\frac{1}{9} - \frac{1}{16}\right) = \frac{7}{16}
\end{aligned}$$

A simpler solution involving no integrals is as follows:
The bulb failure times of interest (1st type-$A$, 2nd type-$B$) may be thought of as the arrival

times of two independent Poisson processes of rate $\lambda_A = 1$ and $\lambda_B = 3$. We may imagine that these two processes were split from a joint Poisson process of rate $\lambda_A + \lambda_B$, where the splitting probabilities for each arrival are $P(A) = \frac{\lambda_A}{\lambda_A+\lambda_B} = 1/4$ to process $A$ and $P(B) = \frac{\lambda_B}{\lambda_A+\lambda_B} = 3/4$ to process $B$. Now we may just focus on whether arrivals to the joint process go to process $A$ or to process $B$. Each arrival to the joint process corresponds to an independent trial. There are two possible outcomes: the arrival is handed to process A with probability $P(A)$ or the arrival is handed to process B with probability $P(B)$. Then our event of interest occurs when either the first arrival goes to A, or the first arrival goes to B followed by the second going to A. So the corresponding probability is

$$P(A \text{ or } BA) = P(A) + P(BA) = P(A) + P(B)P(A) = 7/16$$

(e) Let $V$ be the total period of illumination provided by type-B bulbs while the process is in operation. Let $N$ be the number of light bulbs, out of the first 12, that are of type-B. Let $X_i$ be the period of illumination from the $i$th type-B bulb. We then have $V = Y_1 + \cdots Y_N$. Note that $N$ is a binomial random variable, with parameters $n = 12$ and $p = 1/2$, so that

$$\mathbf{E}[N] = 6, \qquad \text{var}(N) = 12 \cdot \frac{1}{2} \cdot \frac{1}{2} = 3.$$

Furthermore, $\mathbf{E}[X_i] = 1/3$ and $\text{var}(X_i) = 1/9$. Using the formulas for the mean and variance of the sum of a random number of random variables, we obtain

$$\mathbf{E}[V] = \mathbf{E}[N]\mathbf{E}[X_i] = 2,$$

and

$$\text{var}(V) = \text{var}(X_i)\mathbf{E}[N] + (\mathbf{E}[X_i])^2\text{var}(N) = \frac{1}{9} \cdot 6 + \frac{1}{9} \cdot 3 = 1$$

(f) Using the notation in parts (a)-(c), and the result of part (c), we have

$$\begin{aligned}\mathbf{E}[T|D] &= t + \mathbf{E}[T - t|D \cap A]\mathbf{P}(A|D) + \mathbf{E}[T - t|D \cap B]\mathbf{P}(B|D) \\ &= t + 1 \cdot \frac{1}{1 + e^{-2t}} + \frac{1}{3}\left(1 - \frac{1}{1 + e^{-2t}}\right) \\ &= t + \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{1 + e^{-2t}}.\end{aligned}$$

2. (a) The total arrival process corresponds to the merging of two independent Poisson processes, and is therefore Poisson with rate $\lambda = \lambda_A + \lambda_B = 7$. Thus, the number $N$ of jobs that arrive in a given three-minute interval is a Poisson random variable, with $\mathbf{E}[N] = 3\lambda = 21$, $\text{var}(N) = 21$, and PMF

$$p_N(n) = \frac{(21)^n e^{-21}}{n!}, \qquad n = 0, 1, 2, \ldots.$$

(b) Each of these 10 jobs has probability $\lambda_A/(\lambda_A + \lambda_B) = 3/7$ of being type A, independently of the others. Thus, the binomial PMF applies and the desired probability is equal to

$$\binom{10}{3}\left(\frac{3}{7}\right)^3\left(\frac{4}{7}\right)^7$$

(c) Each future arrival is of type A with probability $\lambda_A/(\lambda_A + \lambda_B) = 3/7$ of being type A, independently of the others. Thus, the number $K$ of arrivals until the first type A arrival is geometric with parameter 3/7. The number of type B arrivals before the first type A arrival is equal to $K - 1$, and its PMF is similar to a geometric, except that it is shifted by one unit to the left. In particular,

$$p_K(k) = \left(\frac{3}{7}\right)\left(\frac{4}{7}\right)^k, \qquad k = 0, 1, 2, \ldots.$$

3. The event $\{X < Y < Z\}$ can be expressed as $\{X < \min\{Y, Z\}\} \cap \{Y < Z\}$. Let $Y$ and $Z$ be the 1st arrival times of two independent Poisson processes with rates $\mu$ and $\nu$. By merging the two processes, it should be clear that $Y < Z$ if and only if the first arrival of the merged process comes from the original process with rate $\mu$, and thus

$$\mathbf{P}(Y < Z) = \frac{\mu}{\mu + \nu}\ .$$

Let $X$ be the 1st arrival time of a third independent Poisson process with rate $\lambda$. Now $\{X < \min\{Y, Z\}\}$ if and only if the first arrival of the Poisson process obtained by merging the two processes with rates $\lambda$ and $\mu + \nu$ comes from the original process with rate $\lambda$, and thus

$$\mathbf{P}(X < \min\{Y, Z\}) = \frac{\lambda}{\lambda + \mu + \nu}\ .$$

Note that the event $\{X < \min\{Y, Z\}\}$ is independent of the event $\{Y < Z\}$, as the time of the first arrival of the merged process with rate $\mu + \nu$ is independent of whether that first arrival comes from the process with rate $\mu$ or the process with rate $\nu$. Hence,

$$\begin{aligned} \mathbf{P}(X < Y < Z) &= \mathbf{P}(X < \min\{Y, Z\}) \cdot \mathbf{P}(Y < Z) \\ &= \frac{\lambda\mu}{(\lambda + \mu + \nu)(\mu + \nu)}. \end{aligned}$$