Recitation 11 Solutions
October 14, 2010

1. We need to apply the version of Bayes rule for a discrete random variable conditioned on a continuous random variable:

$$p_{X|Z}(x \mid z) = \frac{p_X(x) f_{Z|X}(z \mid x)}{f_Z(z)} = \frac{p_X(x) f_{Z|X}(z \mid x)}{\sum_{k=0}^{1} p_X(k) f_{Z|X}(z \mid k)}.$$

Specifically,

$$\begin{aligned}
\mathbf{P}(X = 1 \mid Z = z) &= p_{X|Z}(1 \mid z) = \frac{p_X(1) f_{Z|X}(z \mid 1)}{\sum_{k=0}^{1} p_X(k) f_{Z|X}(z \mid k)} \\
&= \frac{p\frac{1}{2}\lambda e^{-\lambda|z-1|}}{(1-p)\frac{1}{2}\lambda e^{-\lambda|z+1|} + p\frac{1}{2}\lambda e^{-\lambda|z-1|}} \\
&= \frac{pe^{-\lambda|z-1|}}{(1-p)e^{-\lambda|z+1|} + pe^{-\lambda|z-1|}} \\
&= \frac{pe^{-\lambda|z-1|}}{(1-p)e^{-\lambda|z+1|} + pe^{-\lambda|z-1|}} \cdot \frac{e^{\lambda|z-1|}}{e^{\lambda|z-1|}} \\
&= \frac{p}{(1-p)e^{-\lambda(|z+1|-|z-1|)} + p}
\end{aligned}$$

The final manipulations are to ease interpretations for $p \to 0^+$, $p \to 1^-$, $\lambda \to 0^+$, and $\lambda \to \infty$. Easily

$$\lim_{p \to 0^+} \mathbf{P}(X = 1 \mid Z = z) = 0 \qquad \text{and} \qquad \lim_{p \to 1^-} \mathbf{P}(X = 1 \mid Z = z) = 1;$$

these make sense because the observation z should become unimportant when value of X becomes certain without it. Next,

$$\lim_{\lambda \to 0^+} \mathbf{P}(X = 1 \mid Z = z) = p,$$

which makes sense because the distribution of Y becomes very flat as $\lambda \to 0^+$, making the observation uninformative. Finally,

$$\lim_{\lambda \to \infty} \mathbf{P}(X = 1 \mid Z = z) = \begin{cases} 1, & \text{if } |z+1| > |z-1|, \\ 0, & \text{if } |z+1| < |z-1|, \end{cases} = \begin{cases} 1, & \text{if } z > 0, \\ 0, & \text{if } z < 0; \end{cases}$$

this makes sense because $\lambda \to \infty$ makes the Y negligible.

2. We need to apply the version of Bayes rule for a continuous random variable conditioned on a discrete random variable:

$$f_{Q|X}(q \mid x) = \frac{f_Q(q) p_{X|Q}(x \mid q)}{p_X(x)} = \frac{f_Q(q) p_{X|Q}(x \mid q)}{\int_0^1 f_Q(q) p_{X|Q}(x \mid q)\, dq}.$$

For $x = 0$ and $q \in [0, 1]$,

$$\begin{aligned}
f_{Q|X}(q \mid 0) &= \frac{f_Q(q) p_{X|Q}(0 \mid q)}{\int_0^1 f_Q(q) p_{X|Q}(0 \mid q)\, dq} = \frac{6q(1-q)\cdot(1-q)}{\int_0^1 6q(1-q)(1-q)\, dq} \\
&= \frac{6q(1-q)\cdot(1-q)}{1/2} = 12q(1-q)^2.
\end{aligned}$$

For $x = 1$ and $q \in [0, 1]$,

$$\begin{aligned} f_{Q|X}(q \mid 1) &= \frac{f_Q(q) p_{X|Q}(1 \mid q)}{\int_0^1 f_Q(q) p_{X|Q}(1 \mid q)\, dq} = \frac{6q(1-q) \cdot q}{\int_0^1 6q(1-q)q\, dq} \\ &= \frac{6q(1-q) \cdot q}{1/2} = 12q^2(1-q). \end{aligned}$$

The distributions $f_Q(q)$, $f_{Q|X}(q \mid 0)$, and $f_{Q|X}(q \mid 1)$ are all in the family of *beta distributions*, which arise again in Chapter 8.

3. Because of the definition of g, the random variable Y takes on only nonnegative values. Thus $f_Y(y) = 0$ for any negative y. For $y > 0$,

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \leq y) \\ &= \mathbf{P}\left(X \in [-y, 0]\right) + \mathbf{P}\left(X \in (0, y^2]\right) \\ &= \left(F_X(0) - F_X(-y)\right) + \left(F_X(y^2) - F_X(0)\right) \\ &= F_X(y^2) - F_X(-y). \end{aligned}$$

Taking the derivative of $F_Y(y)$ (and using the chain rule),

$$\begin{aligned} f_Y(y) &= 2y f_X(y^2) + f_X(-y) \\ &= \frac{1}{\sqrt{2\pi}} \left(2y e^{-y^4/2} + e^{-y^2/2} \right). \end{aligned}$$