Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 7 Solutions
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1. See the textbook, Problem 2.35, page 130.

2. (a)

$$\begin{aligned} p_X(1) &= \mathbf{P}(X=1, Y=1) + \mathbf{P}(X=1, Y=2) + \mathbf{P}(X=1, Y=3) \\ &= 1/12 + 2/12 + 1/12 = 1/3 \end{aligned}$$

   (b) The solution is a sketch of the following conditional PMF:

$$p_{Y|X}(y \mid 1) = \frac{p_{Y,X}(y,1)}{p_X(1)} = \begin{cases} 1/4, & \text{if } y = 1, \\ 1/2, & \text{if } y = 2, \\ 1/4, & \text{if } y = 3, \\ 0, & \text{otherwise.} \end{cases}$$

   (c) $\mathbf{E}[Y \mid X = 1] = \sum_{y=1}^{3} y\, p_{Y|X}(y \mid 1) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} = 2$

   (d) Assume that X and Y are independent. Because $p_{X,Y}(3,1) = 0$ and $p_Y(1) = 1/4$, $p_X(3)$ must equal zero. This further implies $p_{X,Y}(3,2) = 0$ and $p_{X,Y}(3,3) = 0$. All the remaining probability mass must go to $(X,Y) = (2,2)$, making $p_{X,Y}(2,2) = 5/12$, $p_X(2) = 8/12$, and $p_Y(2) = 7/12$. However, $p_{X,Y}(2,2) \neq p_X(2) \cdot p_Y(2)$, contradicting the assumption; thus X and Y are not independent.

   A simpler explanation uses only two X values and two Y values for which all four (X,Y) pairs have specified probabilities. Note that if X and Y are independent, then $p_{X,Y}(1,3)/p_{X,Y}(1,1)$ and $p_{X,Y}(2,3)/p_{X,Y}(2,1)$ must be equal because they must both equal $p_Y(3)/p_Y(1)$. This necessary equality does not hold, so X and Y are not independent.

   (e) Knowing that X and Y are conditionally independent given B, we must have

$$\frac{p_{X,Y}(1,1)}{p_{X,Y}(1,2)} = \frac{p_{X,Y}(2,1)}{p_{X,Y}(2,2)}$$

   since the (X,Y) pairs in the equality are all in B. Thus

$$p_{X,Y}(2,2) = \frac{p_{X,Y}(1,2)p_{X,Y}(2,1)}{p_{X,Y}(1,1)} = \frac{(2/12)(2/12)}{1/12} = \frac{4}{12} = \frac{1}{3}.$$

   (f) Since $\mathbf{P}(B) = 9/12 = 3/4$, we normalize to obtain $p_{X,Y|B}(2,2) = \frac{p_{X,Y}(2,2)}{\mathbf{P}(B)} = 4/9$.

3. See the textbook, Problem 2.33, page 128.

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