Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

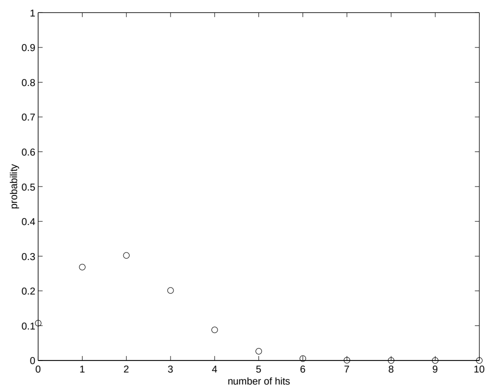
Recitation 5 Solutions
September 23, 2010

1. (a) See derivation in textbook pp. 84-85.
   (b) See derivation in textbook p. 86.
   (c) See derivation in textbook p. 87.

2. (a) X is a Binomial random variable with $n = 10$, $p = 0.2$. Therefore,

$$p_X(k) = \binom{10}{k} 0.2^k \, 0.8^{10-k}, \qquad \text{for } k = 0, \ldots, 10$$

and $p_X(k) = 0$ otherwise.



   (b) $\mathbf{P}(\text{No hits}) = p_X(0) = (0.8)^{10} = \boxed{0.1074}$
   (c) $\mathbf{P}(\text{More hists than misses}) = \sum_{k=6}^{10} p_X(k) = \sum_{k=6}^{10} \binom{10}{k} 0.2^k \, 0.8^{10-k} = \boxed{0.0064}$
   (d) Since X is a Binomial random variable,

$$\mathbf{E}[X] = 10 \cdot 0.2 = \boxed{2} \qquad \text{var}(X) = 10 \cdot 0.2 \cdot 0.8 = \boxed{1.6}$$

   (e) $Y = 2X - 3$, and therefore

$$\mathbf{E}[Y] = 2\mathbf{E}[X] - 3 = \boxed{1} \qquad \text{var}(Y) = 4\text{var}(X) = \boxed{6.4}$$

   (f) $Z = X^2$, and therefore

$$\mathbf{E}[Z] = \mathbf{E}[X^2] = (\mathbf{E}[X])^2 + \text{var}(X) = \boxed{5.6}$$

3. (a) We expect $\mathbf{E}[X]$ to be higher than $\mathbf{E}[Y]$ since if we choose the student, we are more likely to pick a bus with more students.
   (b) To solve this problem formally, we first compute the PMF of each random variable and then compute their expectations.

$$p_X(x) = \begin{cases} 40/148 & x = 40 \\ 33/148 & x = 33 \\ 25/148 & x = 25 \\ 50/148 & x = 50 \\ 0 & \text{otherwise.} \end{cases}$$

and $\mathbf{E}[X] = 40\,\frac{40}{148} + 33\,\frac{33}{148} + 25\,\frac{25}{148} + 50\,\frac{50}{148} = 39.28$

$$p_Y(y) = \begin{cases} 1/4 & y = 40, 33, 25, 50 \\ 0 & \text{otherwise.} \end{cases}$$

and $\mathbf{E}[Y] = 40\,\frac{1}{4} + 33\,\frac{1}{4} + 25\,\frac{1}{4} + 50\,\frac{1}{4} = 37$
Clearly, $\mathbf{E}[X] > \mathbf{E}[Y]$.

4. The expected value of the gain for a single game is infinite since if X is your gain, then

$$\sum_{k=1}^{\infty} 2^k \cdot 2^{-k} = \sum_{k=1}^{\infty} 1 = \infty$$

Thus if you are faced with the choice of playing for given fee f or not playing at all,and your objective is to make the choice that maximizes your expected net gain, you would be willing to pay any value of f. However, this is in strong disagreement with the behavior of individuals. In fact experiments have shown that most people are willing to pay only about \$20 to \$30 to play the game. The discrepancy is due to a presumption that the amount one is willing to pay is determined by the expected gain. However, expected gain does not take into account a persons attitude towards risk taking.

Below are histograms showing the payout results for various numbers of simulations of this game: