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6.02 Practice Problems: Noise & Bit Errors

Problem 1.

Suppose the bit detection sample at the receiver is `V + noise` volts when the sample corresponds to a transmitted '1', and `0.0 + noise` volts when the sample corresponds to a transmitted '0', where `noise` is a zero-mean Normal(Gaussian) random variable with standard deviation σ_{NOISE}.

A. If the transmitter is equally likely to send '0"s or '1"s, and `V/2` volts is used as the threshold for deciding whether the received bit is a '0' or a '1', give an expression for the bit-error rate (BER) in terms of the erfc function and σ_{NOISE}.

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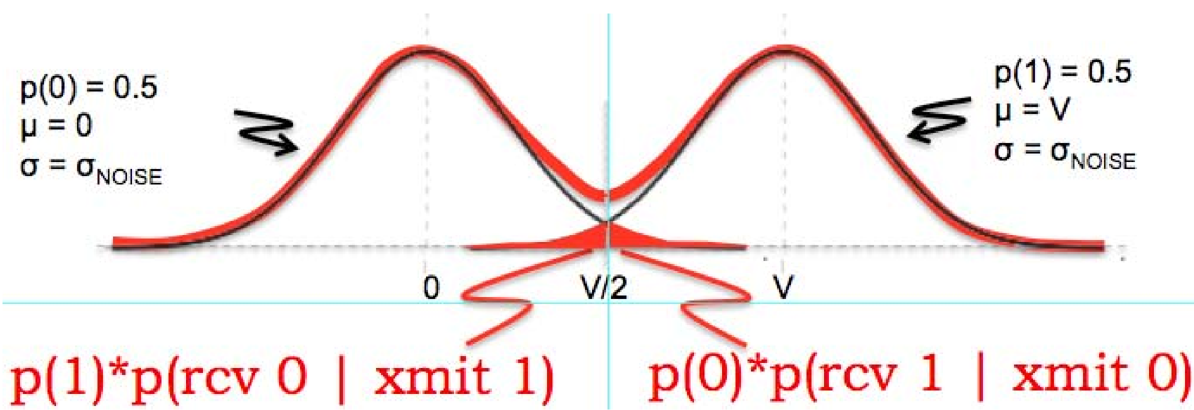
Here's a plot of the PDF for the received signal where the red-shaded areas correspond to the probabilities of receiving a bit in error.



so the bit-error rate is given by 0.5*erfc(V/(sqrt(8)*sigma)). Note that sigma = sqrt(N0/2) using the definition of N0. This formula is related to the 0.5*sqrt(E_s/N0) from Chapter 5 and the lecture; E_s in our case is V*V/4.

B. Suppose the transmitter is equally likely to send zeros or ones and uses zero volt samples to represent a '0' and one volt samples to represent a '1'. If the receiver uses 0.5 volts as the threshold for deciding bit value, for what value of σ_{NOISE} is the probability of a bit error approximately equal to 1/5?

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From part (A), 0.5*erfc(1/(sqrt(8)*σ_{NOISE})) = 0.2, which gives us σ_{NOISE} = 0.594.

C. Will your answer for σ_{NOISE} in part (B) change if the threshold used by the receiver is shifted to 0.6 volts? Do not try to determine σ_{NOISE}, but justify your answer.

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If move Vth higher to 0.6V, we'll be decreasing prob(rcv1|xmit0) and increasing prob(rcv0|xmit1). Considering the shape of the Gaussian PDF, the decrease will be noticeably smaller than the increase, so we'd expect BER to increase for a given σ_{NOISE}. Thus to keep BER = 1/5, we'd need to *decrease* our estimate for σ_{NOISE}. One can also work out the same result with some algebra; we saw in Chapter 6 how picking the mid-point threshold minimizes the bit error rate.

D. Will your answer for σ_{NOISE} in part (B) change if the transmitter is twice as likely to send ones as zeros, but the receiver still uses a threshold of 0.5 volts? Do not try to determine σ_{NOISE}, but justify your answer.

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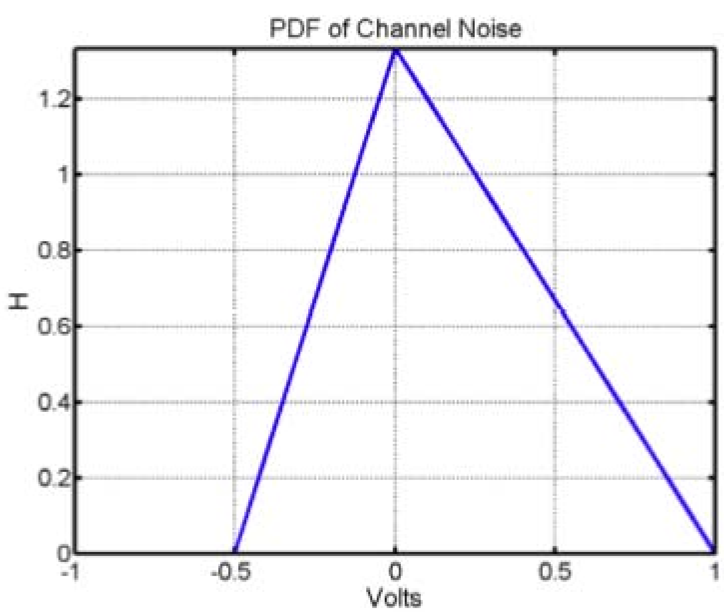
If we change the probabilities of transmission but keep the same digitization threshold, the various parts of the BER equation in (A) are weighted differently (to reflect the different transmission probabilities), but the total BER remains unchanged. This question is essentially the same as one on PSet 3.

Problem 2.

Messages are transmitted along a noisy channel using the following protocol: a "0" bit is transmitted as -0.5 Volt and a "1" bit as 0.5 Volt. The PDF of the total noise added by the channel, H, is shown below. It is not a Gaussian.



a. Compute H(0), the maximum value of H.

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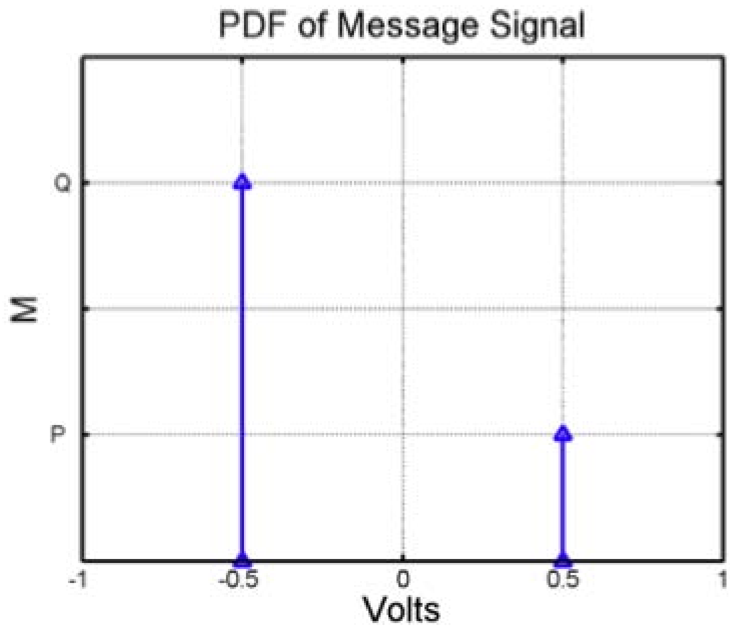
The area under the PDF is 1, so (0.5)*H(0)*(1+0.5) = 1 from which we get H(0) = 4/3.

b. It is known that a "0" bits 3 times as likely to be transmitted as a "1" bit. The PDF of the message signal, M, is shown below. Fill in the values P and Q.



Hide Answer

We know that Q=3P and that P+Q=1, so Q=0.75 and P=.25.

c. If the digitization threshold voltage is 0V, what is the bit error rate?

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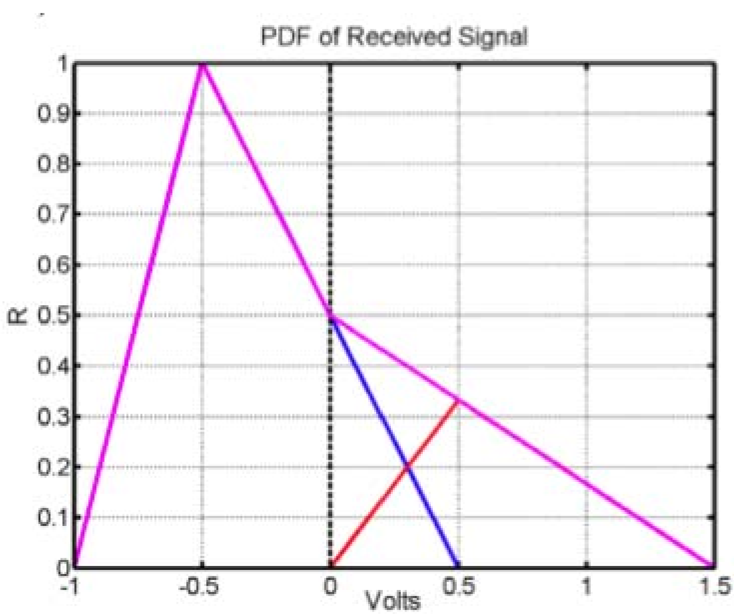
The plot below shown the PDF of the received voltage in magenta. For a threshold voltage of 0, there is only one error possible: a transmitted "0" received as a "1". This error is equal to the area of the triangle formed by the dotted black line and the blue line = 0.5*0.5*0.5 = 0.125.



d. What digitization threshold voltage would minimize the bit error rate?

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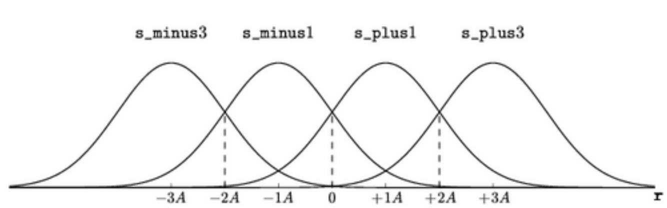
We'll minimize the bit error rate if the threshold voltage is chosen at the voltage where the red and blue lines intersect. By looking at the plot from the previous answer, let the ideal threshold be x and the value of the PDF at the intersection point be y. Then y/x=2/3 and y/(0.5-x)=1, so x = 0.3V.

Problem 3.

Ben Bitdiddle studies the bipolar signaling scheme from 6.02 and decides to extend it to a **4-level signaling scheme**, which he calls Ben's Aggressive Signaling Scheme, or **BASS**. In BASS, the transmitter can send four possible signal levels, or voltages: $(-3A, -A, +A, +3A)$, where A is some positive value. To transmit bits, the sender's mapper maps consecutive pairs of bits to a fixed voltage level that is held for some fixed interval of time, creating a **symbol**. For example, we might map bits "00" to $-3A$, "01" to $-A$, "10" to $+A$, and "11" to $+3A$. Each distinct pair of bits corresponds to a unique symbol. Call these symbols s_minus3, s_minus1, s_plus1, and s_plus3. Each symbol has the same prior probability of being transmitted.

The symbols are transmitted over a channel that has no distortion but does have additive noise, and are sampled at the receiver in the usual way. Assume the samples at the receiver are perturbed from their ideal noise-free values by a zero-mean additive white Gaussian noise (AWGN) process with noise intensity $N_0 = 2\sigma^2$, where σ^2 is the variance of the Gaussian noise on each sample. In the time slot associated with each symbol, the BASS receiver digitizes a selected voltage sample, r, and returns an estimate, s, of the transmitted symbol in that slot, using the following intuitive digitizing rule (written in Python syntax):

```
def digitize(r):
    if r < -2A:
        s = sminus3
    elif r < 0:
        s = sminus1
    elif r < 2A:
         s = splus1
    else: s = splus3
    return s
```



1. The power of a symbol transmission is defined as the square of the voltage level at which the symbol is **transmitted**. What is the **average** power level, P, of a symbol **transmission** in BASS (i.e., the average power dissipated at the transmitter)?

   Hide Answer

$$\frac{(9A^2 + A^2 + A^2 + 9A^2)}{4} = 5A^2$$

Ben wants to calculate the **symbol error rate** for BASS, i.e., the probability that the symbol chosen by the receiver was different from the symbol transmitted. Note: we are **not** interested in the **bit** error rate here. Help Ben calculate the symbol error rate by answering the following questions below from 2 through 8

2. Suppose the sender transmits symbol **s_plus3**. What is the **conditional** symbol error rate given this information; i.e., what is **P(symbolError| s_plus3 sent)** ? Express your answer in terms of A, N_0, and the erfc function, defined as $\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-x^2}\, dx$.

Hide Answer

Using the Gaussian distribution for the noise: the desired probability is:

$$\mathbf{P}(\text{symbol error} \mid \text{s\_plus3 sent}) = \mathbf{P}\left(W \le 2A\right) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{2A} e^{-\frac{(x-3A)^2}{2\sigma^2}}\, dx$$

Substituting $N_0 = 2\sigma^2$, $z = \frac{(x-3A)}{\sqrt{N_0}}$, and $dz = \frac{1}{\sqrt{N_0}}\, dx$ into the above equation, we obtain:

$$\mathbf{P}\left(\text{symbol error} \mid \text{s\_plus3 sent}\right) = \frac{1}{\sqrt{\pi N_0}}\int_{-\infty}^{2A} e^{-\frac{(x-3A)^2}{N_0}}\, dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{-A}{\sqrt{N_0}}} e^{-z^2}\, dx.$$

Since the Gaussian distribution is symmetric, this becomes:

$$\begin{aligned}\mathbf{P}(\text{symbol error} \mid \text{s\_plus3 sent}) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{-A}{\sqrt{N_0}}} e^{-z^2}\, dx \\ &= \frac{1}{\sqrt{\pi}}\int_{\frac{A}{\sqrt{N_0}}}^{\infty} e^{-z^2}\, dx \\ &= \frac{1}{2}\,\mathrm{erfc}\left(\frac{A}{\sqrt{N_0}}\right)\end{aligned}$$

3. Now suppose the sender transmits symbol **s_plus1**. What is the **conditional** symbol error rate given this information, in terms of A, N_0, and the erfc function? The conditional symbol error rates for the other two symbols don't need to be calculated separately.

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An error occurs if the received signal is less that 0 or greater than $2A$. Therefore,

$$\begin{aligned}\mathbf{P}(\text{symbol error} \mid \text{s\_plus1 sent}) &= \mathbf{P}(W \le 0) + \mathbf{P}(W \ge 2A) \\ &= \frac{1}{2}\,\mathrm{erfc}\left(\frac{A}{\sqrt{N_0}}\right) + \frac{1}{2}\,\mathrm{erfc}\left(\frac{A}{\sqrt{N_0}}\right) \\ &= \mathrm{erfc}\left(\frac{A}{\sqrt{N_0}}\right)\end{aligned}$$

where we used the results from problem 8.

4. The symbol error rate when the sender transmits symbol **s_minus3** is the same as the symbol error rate of which of these symbols?

    1. s_minus1.
    2. s_plus1.

3. s_plus3.

Hide Answer

s_plus3 because the Gaussian distribution is symmetric and the distance from the mean of s_minus3 to the error threshold is equal to the distance from the mean of s_plus3 to its error threshold.

5. The symbol error rate when the sender transmits symbol **s_minus1** is the same as the symbol error rate of which of these symbols?
   1. s_minus3.
   2. s_plus1.
   3. s_plus3.

Hide Answer

s_plus1 by symmetry.

6. Now suppose the sender transmits symbol **s_minus1**. What is the **conditional** symbol error rate given this information, in terms of A, N_0, and the erfc function? Do not recalculate; use symmetry and your previous answer(s).

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7. Combining your answers to the questions above, what is the symbol error rate in terms of A, N_0, and the erfc function? Recall that all symbols are equally likely to be transmitted.

Hide Answer

$\mathbf{P}(\text{s\_minus3 sent}) = \mathbf{P}(\text{s\_minus1 sent}) = \mathbf{P}(\text{s\_plus1 sent}) = \mathbf{P}(\text{s\_plus3 sent}) = \frac{1}{4}$.

The symbol error rate is:

$$\begin{aligned}
\mathbf{P}(\text{symbol error}) &= \mathbf{P}(\text{s_minus3 sent})\mathbf{P}(\text{symbol error} \mid \text{s_minus3 sent}) \\
&\quad +\mathbf{P}(\text{s_minus1 sent})\mathbf{P}(\text{symbol error} \mid \text{s_minus1 sent}) \\
&\quad +\mathbf{P}(\text{s_plus1 sent})\mathbf{P}(\text{symbol error} \mid \text{s_plus1 sent}) \\
&\quad +\mathbf{P}(\text{s_plus3 sent})\mathbf{P}(\text{symbol error} \mid \text{s_plus3 sent}) \\
&= \frac{1}{4}\left(\frac{1}{2}\operatorname{erfc}\left(\frac{A}{\sqrt{N_0}}\right)\right) + \frac{1}{4}\left(\operatorname{erfc}\left(\frac{A}{\sqrt{N_0}}\right)\right) + \frac{1}{4}\left(\operatorname{erfc}\left(\frac{A}{\sqrt{N_0}}\right)\right) + \frac{1}{4}\left(\frac{1}{2}\operatorname{erfc}\left(\frac{A}{\sqrt{N_0}}\right)\right) \\
&= \frac{3}{4}\operatorname{erfc}\left(\frac{A}{\sqrt{N_0}}\right)
\end{aligned}$$

8. Express the answer to the above question in terms of the signal-to-noise ratio. The average signal power is in the answer to one of the questions above.

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