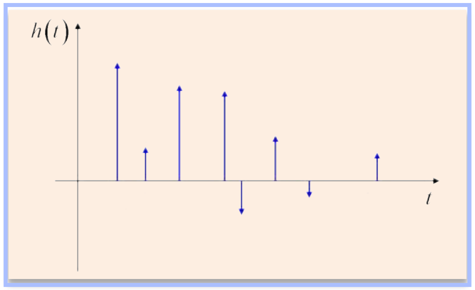
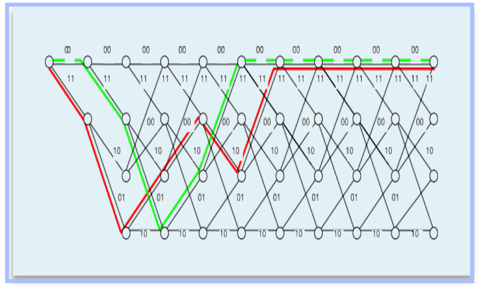
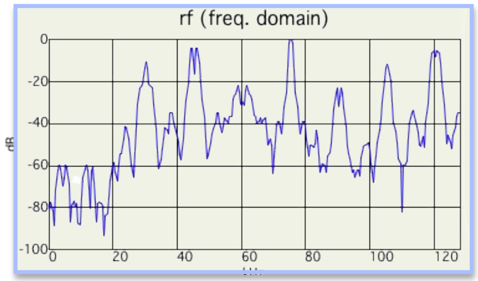
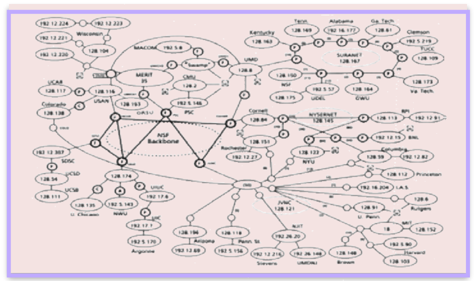








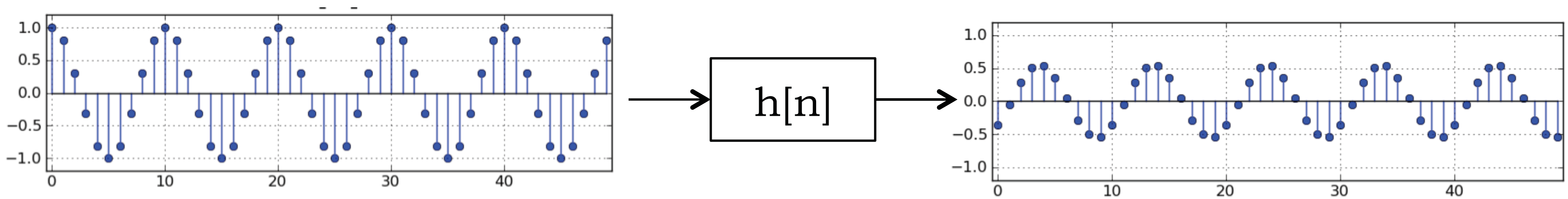
INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2012
Lecture #13

- Frequency response
- Filters
- Spectral content

Sinusoidal Inputs and LTI Systems



A very important property of LTI systems or channels:

> If the input x[n] is a sinusoid of a given amplitude, frequency and phase, the response will be a *sinusoid at the same frequency*, although the amplitude and phase may be altered. The change in amplitude and phase will, in general, depend on the frequency of the input.

Complex Exponentials as "Eigenfunctions" of LTI System

$x[n]=e^{j\Omega n}$ → h[.] → $y[n]=H(\Omega)e^{j\Omega n}$

Eigenfunction: Undergoes only scaling -- by the **frequency response** $H(\Omega)$ in this case:

$$H(\Omega) \equiv \sum_m h[m]e^{-j\Omega m}$$
$$= \sum_m h[m]\cos(\Omega m) - j\sum_m h[m]\sin(\Omega m)$$

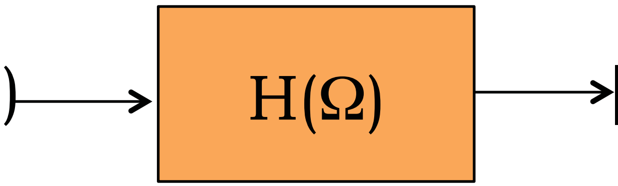
This is an infinite sum in general, but is well behaved if h[.] is absolutely summable, i.e., if the system is stable.

We also call $H(\Omega)$ the **discrete-time Fourier transform (DTFT)** of the time-domain function h[.] --- more on the DTFT later.

From Complex Exponentials to Sinusoids

$\cos(\Omega n) = (e^{j\Omega n} + e^{-j\Omega n}) / 2$

So response to a cosine input is:

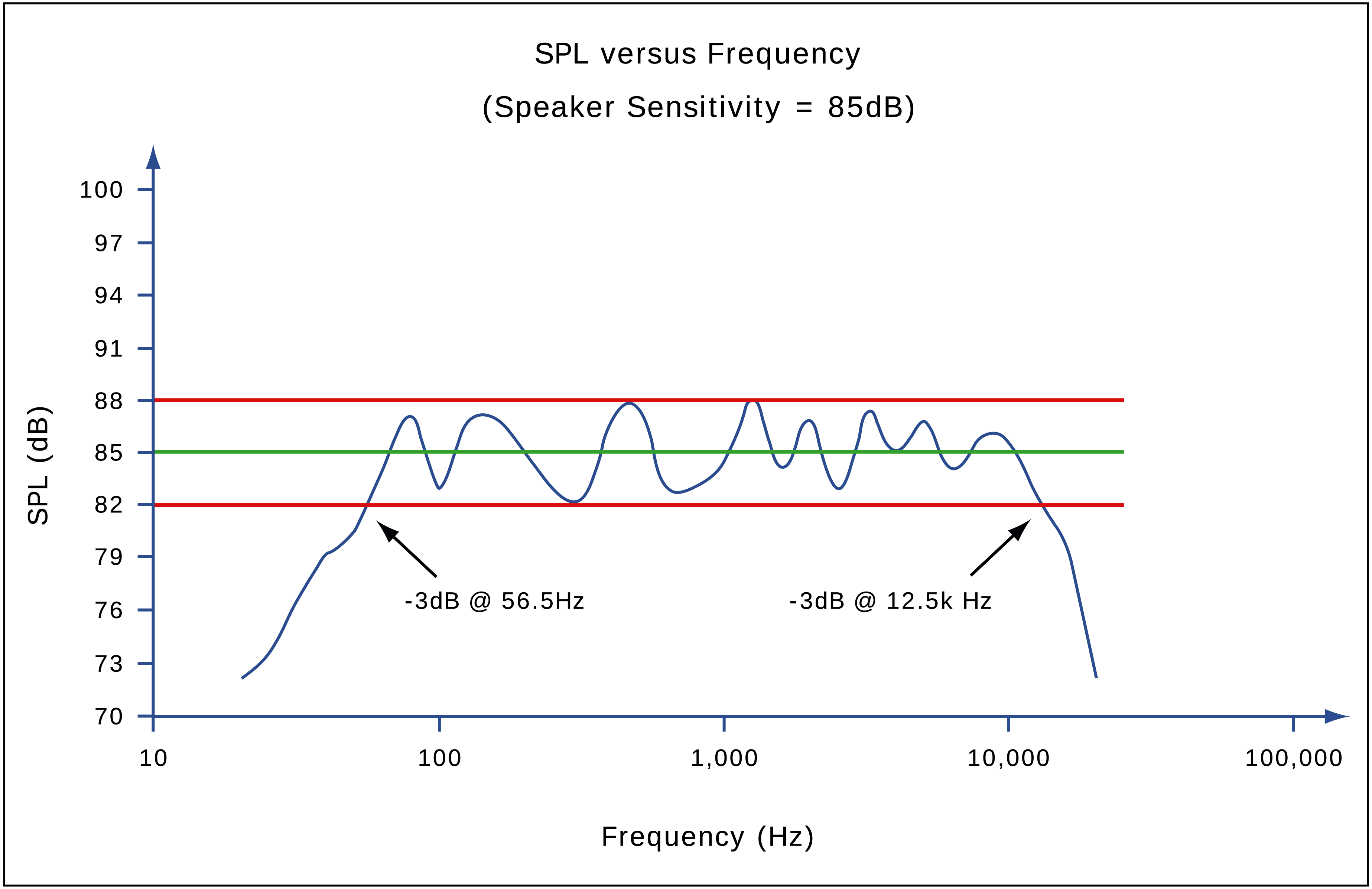
$A\cos(\Omega_0 n + \varnothing_0)$ 

$|\mathbf{H}(\Omega_0)|\, A\cos(\Omega_0 n + \varnothing_0 + \angle \mathbf{H}(\Omega_0))$

(Recall that we only need vary Ω in the interval [–π,π].)

This gives rise to an easy experimental way to determine the frequency response of an LTI system.

Loudspeaker Frequency Response



Spectral Content of Various Sounds

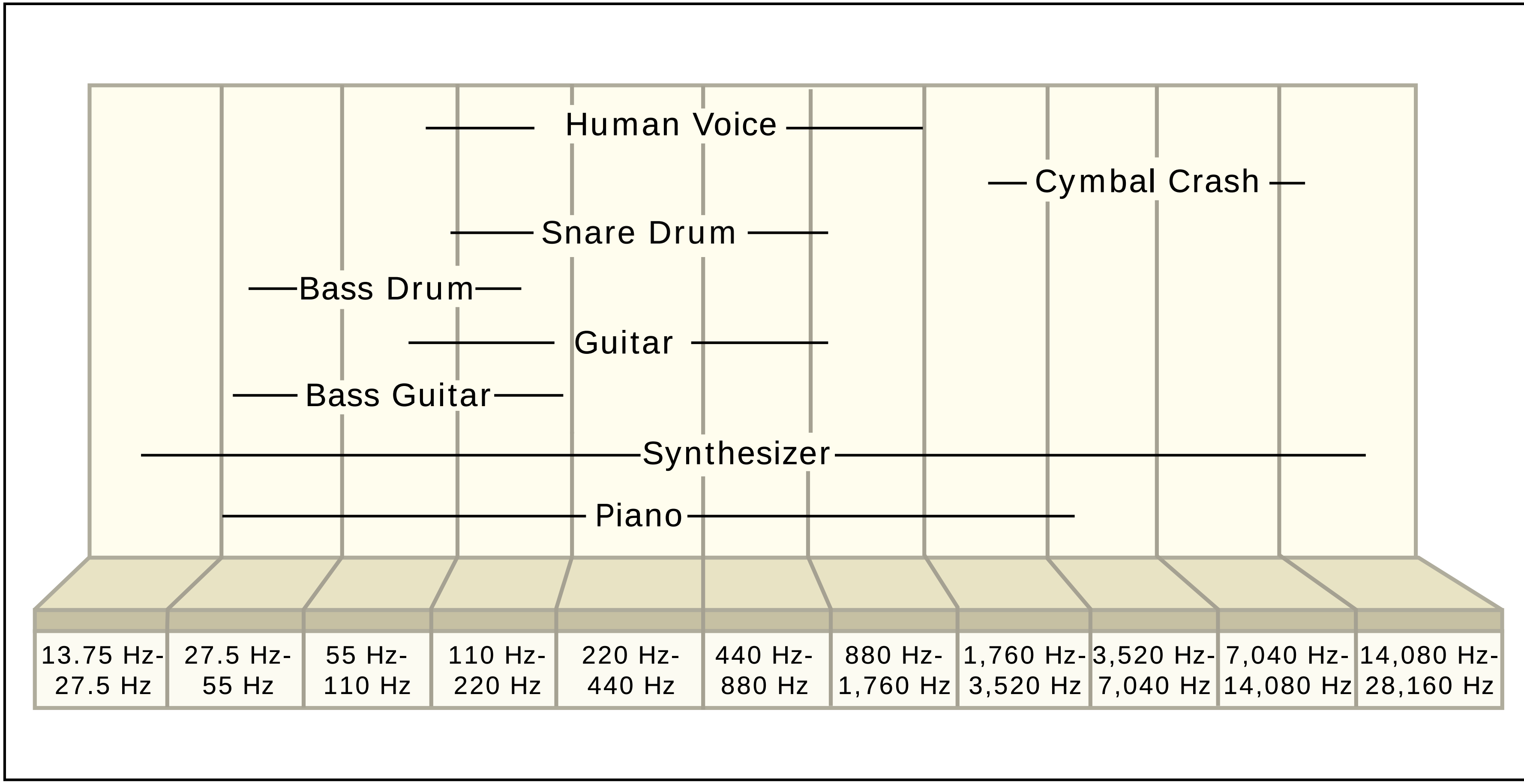


Image by MIT OpenCourseWare.

Connection between CT and DT

The continuous-time (CT) signal

$$x(t) = \cos(\omega t) = \cos(2\pi f t)$$

sampled every T seconds, i.e., at a sampling frequency of $f_s = 1/T$, gives rise to the discrete-time (DT) signal

$$x[n] = x(nT) = \cos(\omega n T) = \cos(\Omega n)$$

So $\Omega = \omega T$

and $\Omega = \pi$ corresponds to $\omega = \pi/T$ or $f = 1/(2T) = f_s/2$

Properties of H(**Ω**)

Repeats periodically on the frequency (Ω) axis, with period 2π, because the input $e^{j\Omega n}$ is the same for Ω that differ by integer multiples of 2π. So only the interval Ω in [−π,π] is of interest!

Properties of $H(\Omega)$

Repeats periodically on the frequency (Ω) axis, with period 2π, because the input $e^{j\Omega n}$ is the same for Ω that differ by integer multiples of 2π. So only the interval Ω in $[-\pi,\pi]$ is of interest!

$\Omega = 0$, i.e., $e^{j\Omega n} = 1$, corresponds to a constant (or "DC", which stands for "direct current", but now just means constant) input, so $H(0)$ is the "DC gain" of the system, i.e., gain for constant inputs.

$H(0) = \sum h[m]$ --- show this from the definition!

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$\Omega = \pi$ or $-\pi$, i.e., $Ae^{j\Omega n}=(-1)^n A$, corresponds to the highest-frequency variation possible for a discrete-time signal, so $H(\pi)=H(-\pi)$ is the high-frequency gain of the system.

$H(\pi) = \sum (-1)^m h[m]$ --- show from definition!

Symmetry Properties of H(Ω)

$$H(\Omega) \equiv \sum_m h[m]e^{-j\Omega m}$$
$$= \sum_m h[m]\cos(\Omega m) - j\sum_m h[m]\sin(\Omega m)$$
$$= C(\Omega) - jS(\Omega)$$

For real h[n]:

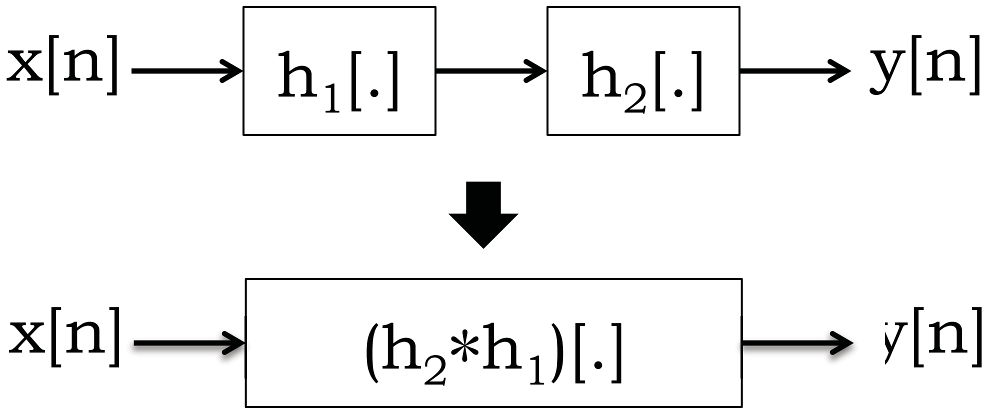
Real part of H(Ω) & **magnitude** are EVEN functions of Ω.

Imaginary part & **phase** are ODD functions of Ω.

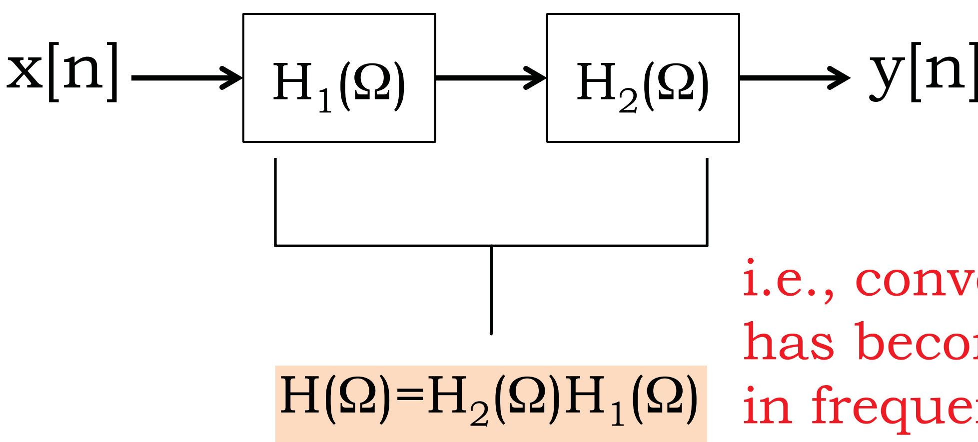
For real and *even* h[n] = h[–n], H(Ω) is purely real.

For real and *odd* h[n] = –h[–n], H(Ω) is purely imaginary.

Convolution in Time <---> Multiplication in Frequency

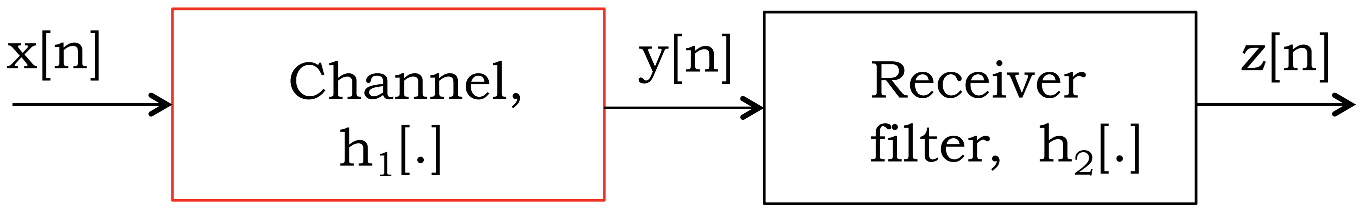


In the frequency domain (i.e., thinking about input-to-output frequency response):



i.e., convolution in time has become multiplication in frequency!

Example: "Deconvolving" Output of Channel with Echo



Suppose channel is LTI with

$$_1[n]=\delta[n]+0.8\delta[n-1]$$

$$H_1(\Omega) = ?? \quad = \boxed{\sum_m h_1[m]e^{-j\Omega m}}$$

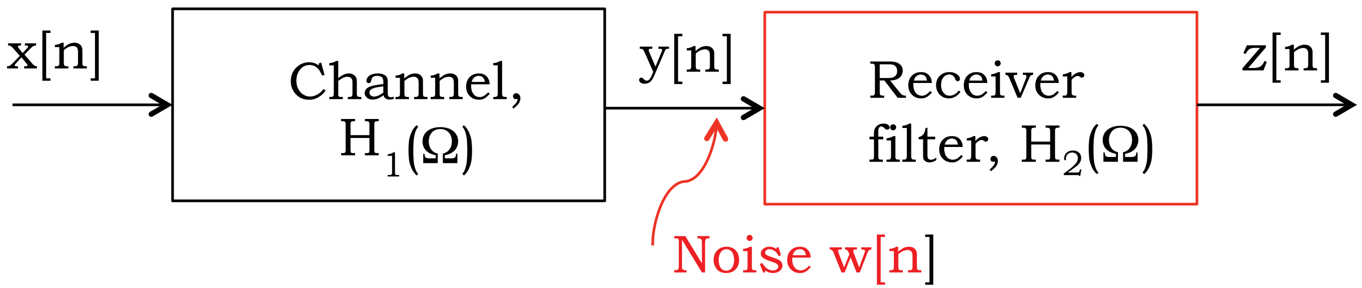
$$= 1+ 0.8e^{-j\Omega} = 1 + 0.8\cos(\Omega) - j0.8\sin(\Omega)$$

So:

$$|H_1(\Omega)| = [1.64 + 1.6\cos(\Omega)]^{1/2}$$ *EVEN function of* Ω;

$$\angle H_1(\Omega) = \arctan\,[-(0.8\sin(\Omega)/[1 + 0.8\cos(\Omega)]$$ *ODD* .

A Frequency-Domain view of Deconvolution



Given $H_1(\Omega)$, what should $H_2(\Omega)$ be, to get z[n]=x[n]?

$$H_2(\Omega) = 1/H_1(\Omega)$$ "Inverse filter"

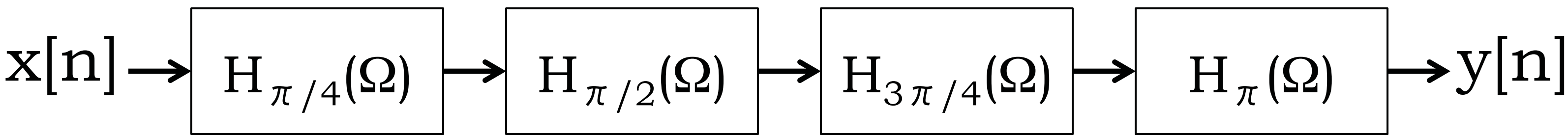
$$= (1/|H_1(\Omega)|).\exp\{-j\angle H_1(\Omega)\}$$

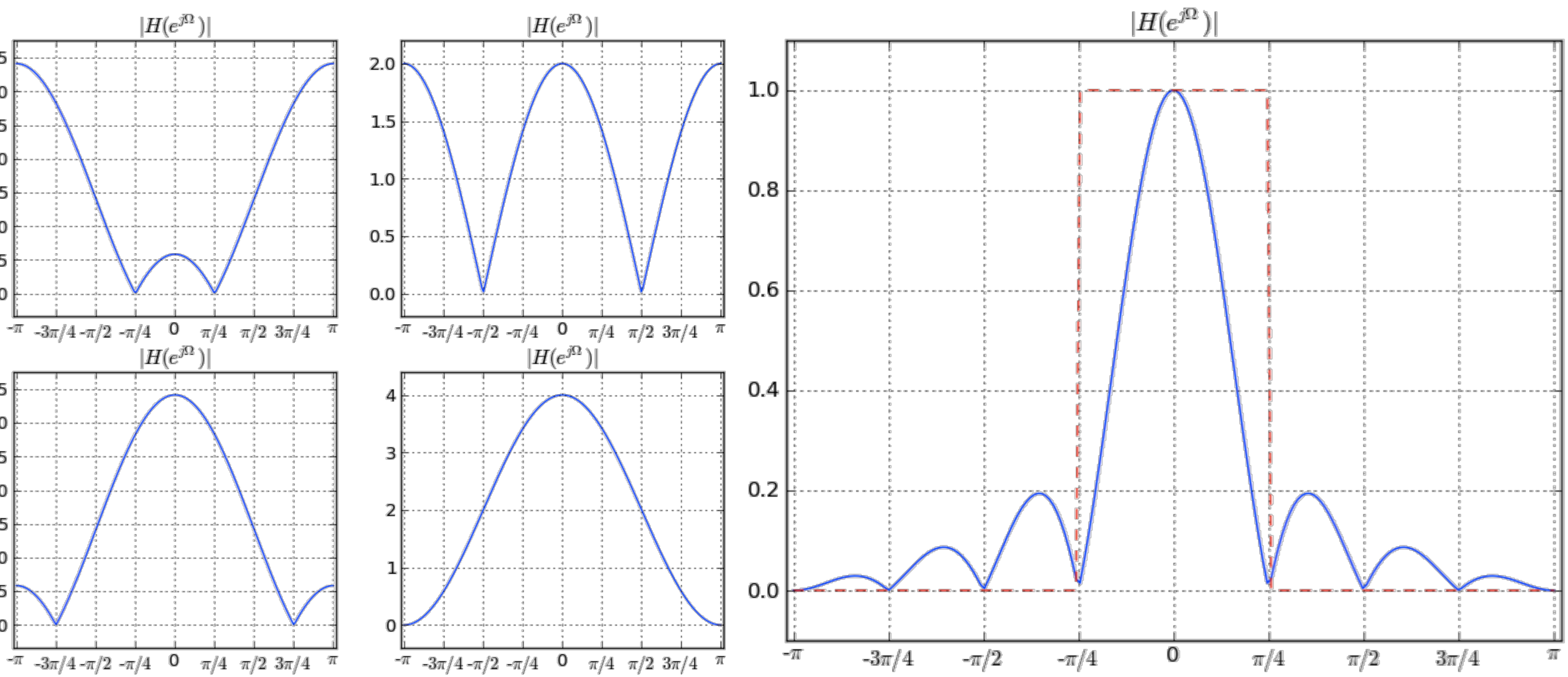
Inverse filter at receiver does very badly in the presence of noise that adds to y[n]:

filter has high gain for noise precisely at frequencies where channel gain $|H_1(\Omega)|$ is low (and channel output is weak)!

A 10-cent Low-pass Filter

Suppose we wanted a low-pass filter with a cutoff frequency of π/4?

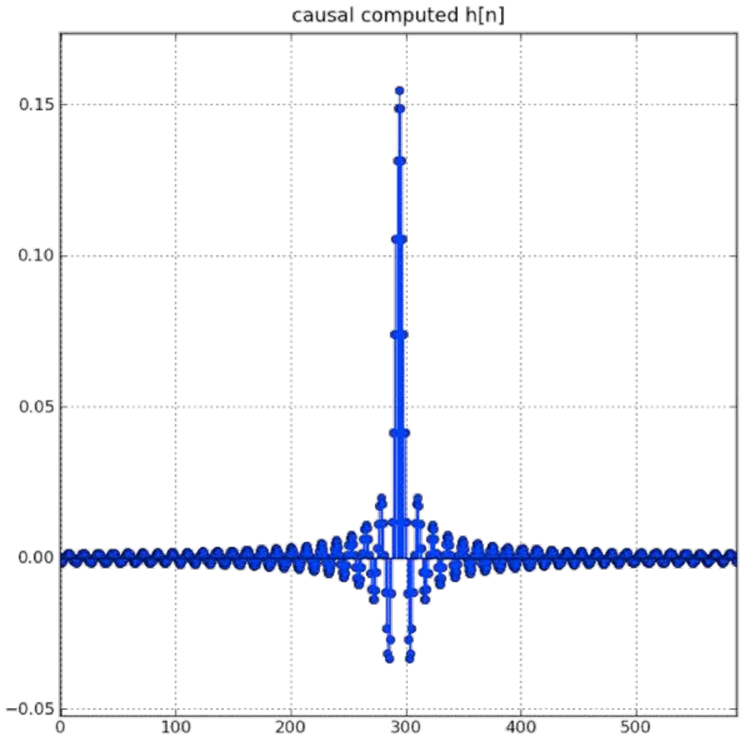


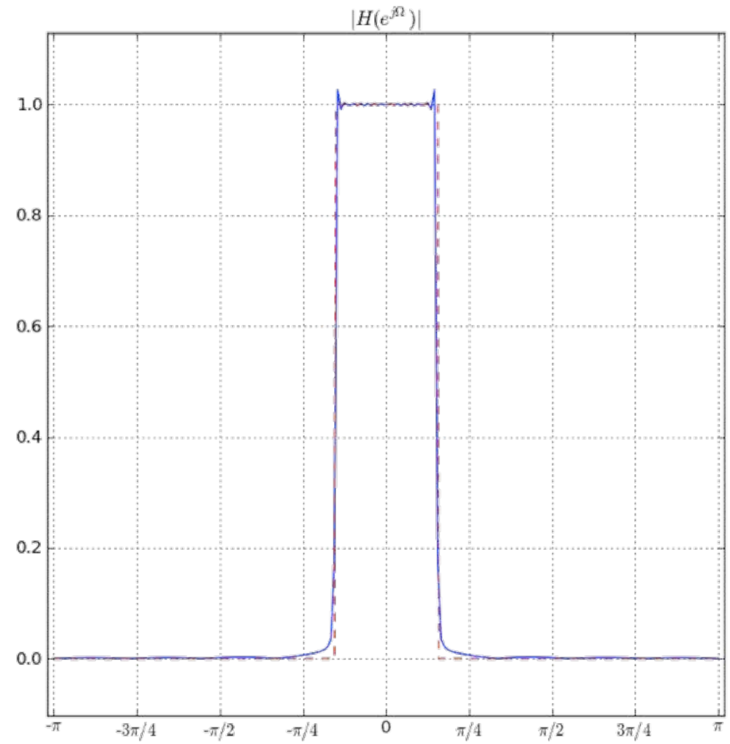


To Get a Filter Section with a Specified Zero-Pair in $H(\Omega)$

- Let $h[0] = h[2] = 1$, $h[1] = \mu$, all other $h[n] = 0$
- Then $H(\Omega) = 1 + \mu e^{-j\Omega} + e^{-j2\Omega} = e^{-j\Omega}(\mu + 2\cos(\Omega))$
- So $|H(\Omega)| = |\mu + 2\cos(\Omega)|$, with zeros at $\pm \arccos(-\mu/2)$

The $4.99 version of a Low-pass Filter, h[n] and H(Ω)





Determining h[n] from H(Ω)

$$H(\Omega) = \sum_{m} h[m] e^{-j\Omega m}$$

Multiply both sides by $e^{j\Omega n}$ and integrate over a (contiguous) 2π interval. Only one term survives!

$$\int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega = \int_{<2\pi>} \sum_{m} h[m] e^{-j\Omega(m-n)} d\Omega$$

$$= 2\pi \cdot h[n]$$

$$\Rightarrow \quad h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega$$

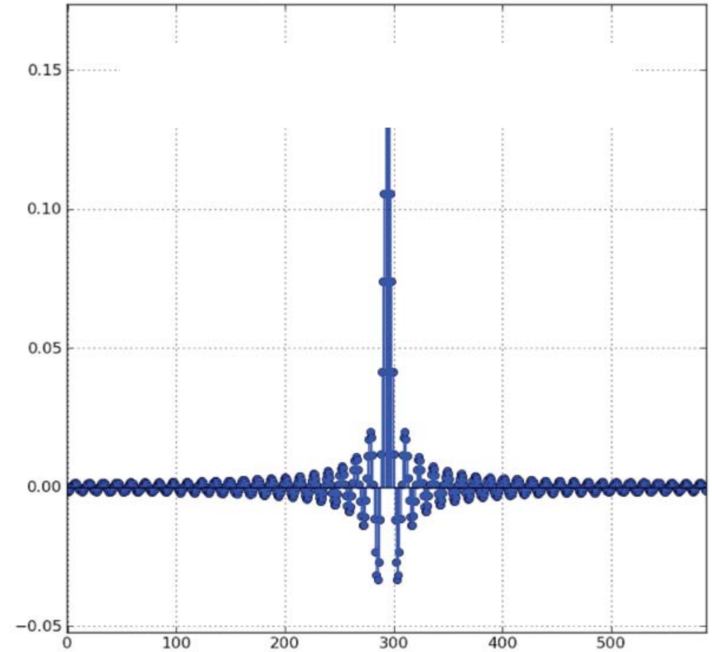
Design ideal lowpass filter with cutoff frequency Ω_C and H(Ω)=1 in passband

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_C}^{\Omega_C} 1 \cdot e^{j\Omega n} d\Omega$$

$$= \frac{\sin(\Omega_C n)}{\pi n}, \quad n \neq 0$$

$$= \Omega_C / \pi \qquad , \quad n = 0$$



DT "sinc" function
(extends to ±∞ in time, falls off only as 1/n))

Exercise: Frequency response of h[n-D]

Given an LTI system with unit sample response h[n] and associated frequency response $H(\Omega)$,

determine the frequency response $H_D(\Omega)$ of an LTI system whose unit sample response is

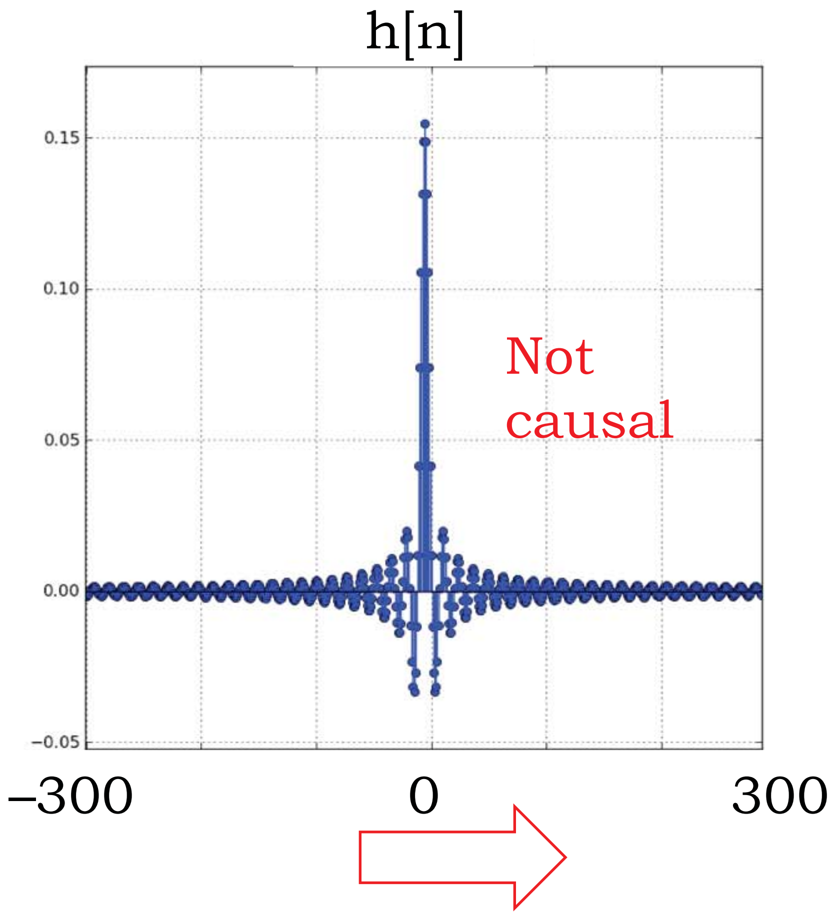
$$h_D[n] = h[n-D].$$

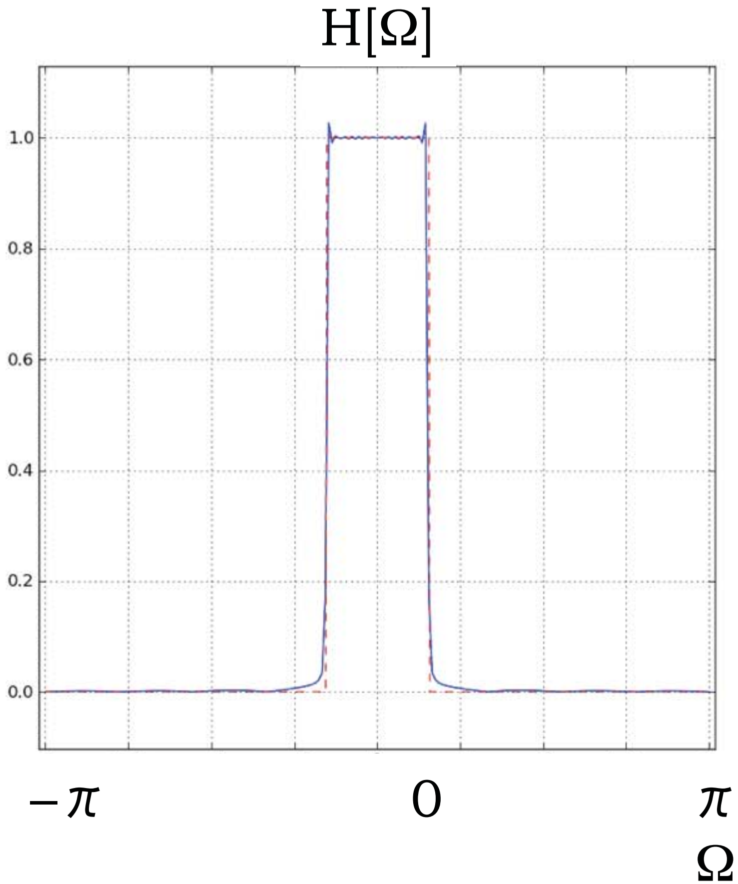
Answer: $H_D(\Omega) = \exp\{-j\Omega D\}.H(\Omega)$

so : $|H_D(\Omega)| = |H(\Omega)|$, i.e., magnitude unchanged

$\angle H_D(\Omega) = -\Omega D + \angle H(\Omega)$, i.e., linear phase term added

e.g.: Approximating an ideal lowpass filter

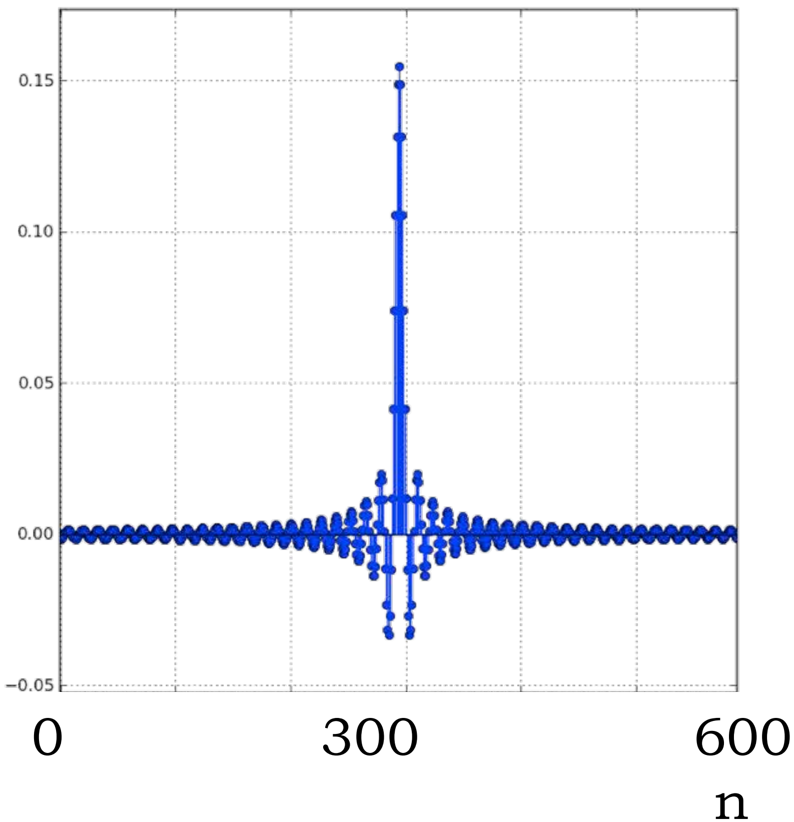




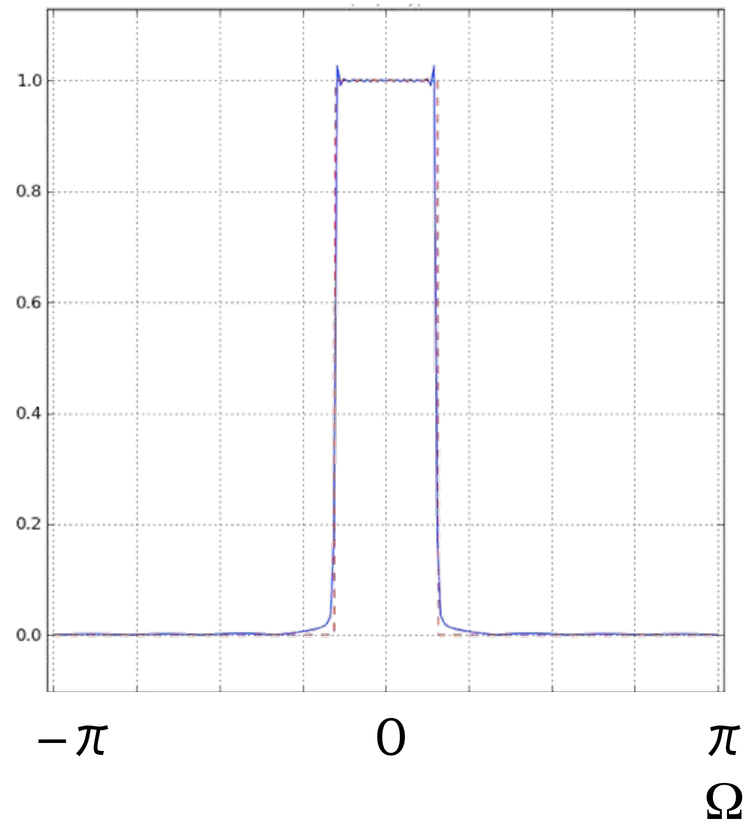
Idea: shift h[n] right to get causal LTI system.
Will the result still be a lowpass filter?

Causal approximation to ideal lowpass filter

$h_C[n] = h[n-300]$



$|H_C[\Omega]|$



Determine $\angle H_C(\Omega)$

DT Fourier Transform (DTFT) for Spectral Representation of General x[n]

If we can write

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega \quad \text{where} \quad H(\Omega) = \sum_n h[n] e^{-j\Omega n}$$

Any contiguous interval of length 2π

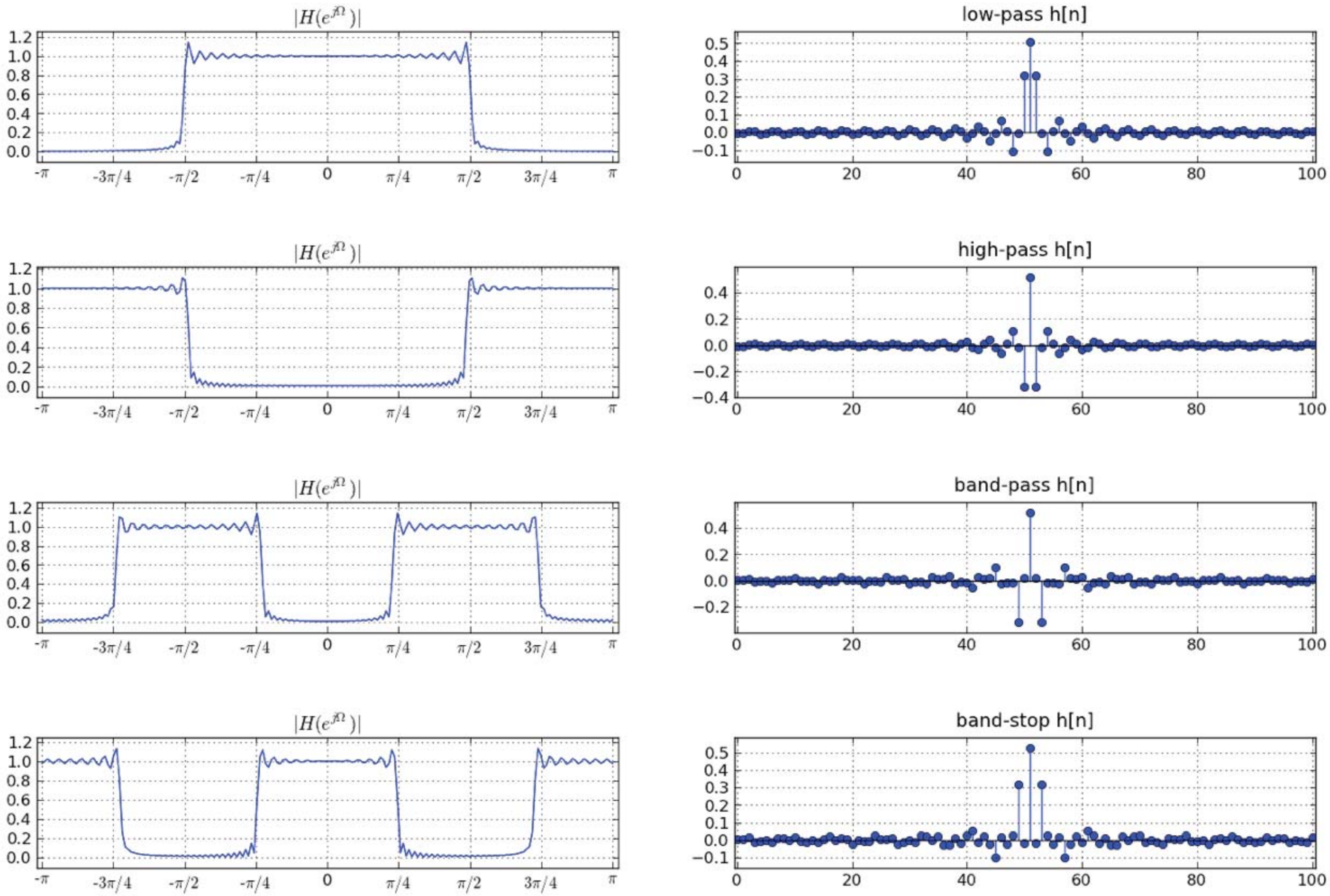
then we can write

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega \quad \text{where} \quad X(\Omega) = \sum_n x[n] e^{-j\Omega n}$$

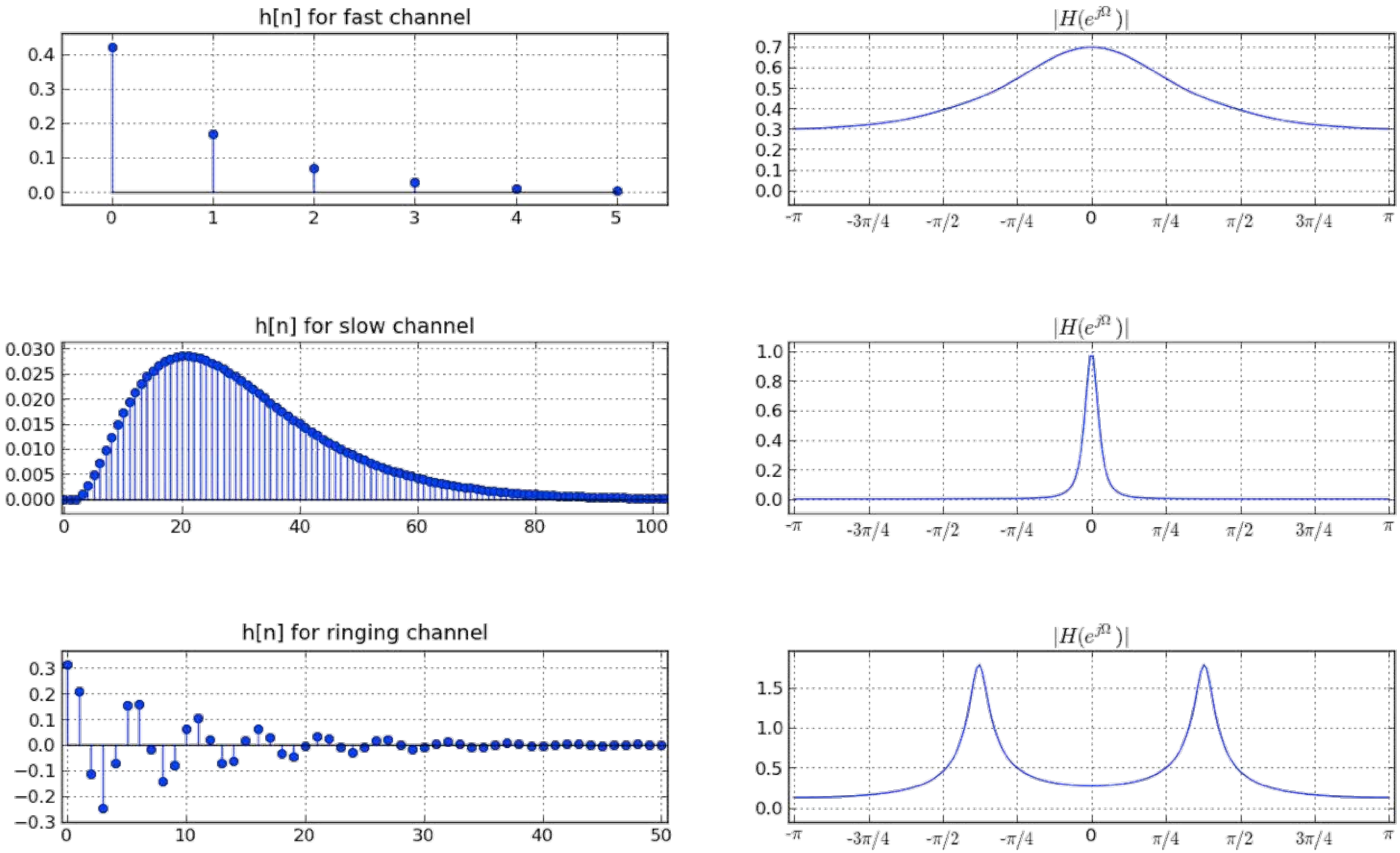
This Fourier representation expresses x[n] as a weighted combination of $e^{j\Omega n}$ for all Ω in $[-\pi, \pi]$.

$X(\Omega_o)d\Omega$ is the **spectral content** of x[n] in the frequency interval $[\Omega_o, \Omega_o + d\Omega]$

Useful Filters



Frequency Response of Channels



6.02 Introduction to EECS II: Digital Communication Systems
Fall 2012