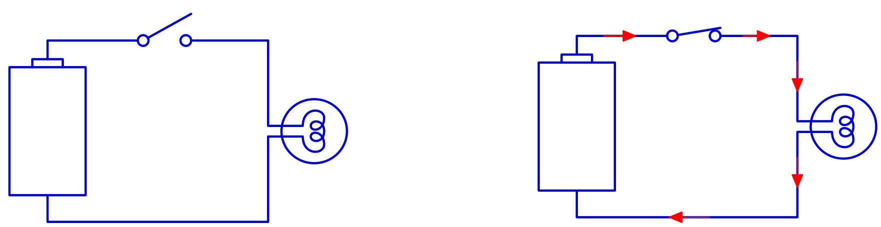
Chapter 6
Circuits

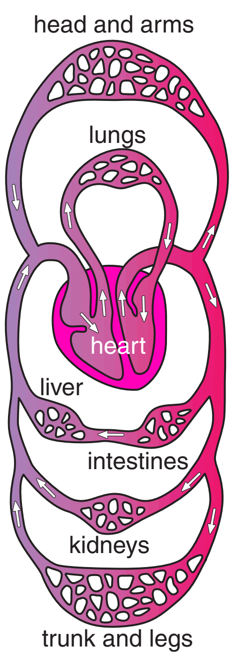
6.1 What is a circuit?

In conventional English, a circuit is a path or route that starts at one place and ultimately returns to that same place. The engineering sense of the word is similar. In electronics, a circuit is a closed path through which electrical currents can flow. The flow of electrical current through a flashlight illustrates the idea.



Three parts of the flashlight comprise the circuit: the battery, which supplies power, a bulb, which produces light when connected to the battery, and a switch, which can either be on (connected) or off (disconnected). The important feature of this circuit is that electrical currents flow through it in a loop, as illustrated in the right part of the figure above.

The rules that govern the flow of electrical current are similar to the rules that govern the flow of certain types of fluids; and it is often helpful to draw analogies between these flows. The following figure illustrates the flow of blood through the human circulatory system.



The right side (from the person's viewpoint) of the heart pumps blood through the lungs and into the left side of the heart, where it is then pumped throughout the body. The heart and associated network of arteries, capillaries, and veins can be thought of and analyzed as a circuit. As with the flow of electrical current in the flashlight example, blood flows through the circulatory system in loops: it starts at one place and ultimately returns to that same place.

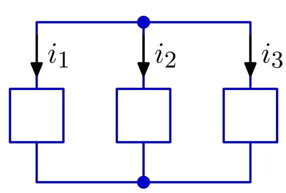
As we learn about circuits, we will point out analogies to fluid dynamics for two reasons. First, we all have developed some intuition for the flow of fluid as a result of our interactions with fluids as a part of everyday experience. We can leverage the similarity between fluid flow and the flow of electrical current to increase our intuition for electrical circuits. Second, the analogy between electrical circuits and fluid dynamics is a simple example of the use of circuits to make models of other phenomena. Such models are widely used in acoustics, hydrodynamics, cellular biophysics, systems biology, neurophysiology, and many other fields.

Electrical circuits are made up of components, such as resistors, capacitors, inductors, and transistors, connected together by wires. You can make arbitrarily amazing, complicated devices by hooking these things up in different ways, but in order to help with analysis and design of circuits, we need a systematic way of understanding how they work.

As usual, we can't comprehend the whole thing at once: it's too hard to analyze the system at the level of individual components, so, again, we're going to build a model in terms of primitives, means of combination, and means of abstraction. The primitives will be the basic components, such as resistors and op-amps; the means of combination is wiring the primitives together into circuits. We'll find that abstraction in circuits is different than in software or LTI systems. You can't think of a circuit as "computing" the voltages on its wires: if you connect it to another circuit, then the voltages are very likely to be different. However, you *can* think of a circuit as enforcing a *constraint* on the voltages and currents that enter and exit it; this constraint will remain true, no matter what else you connect to the circuit.

6.1.1 Electrical circuits

Circuits are made up of *elements* connected by *nodes*. The following circuit has three elements, each represented with a box.



There are two nodes, each indicated by a dot. In an electrical circuit nodes can be thought of as wires[39] that connect a component.

Voltage is a difference in electrical potential between two different points in a circuit. We will often pick some point in a circuit and say that it is "ground" or has voltage 0. Now, every other

[39] Wire is typically made from metals through which electrons can flow to produce electrical currents. Generally, the strong forces of repulsion between electrons prevent their accumulation in wires. However, thinking about nodes as wires is only an approximation, because physical wires can have more complicated electrical properties.

point has a voltage defined with respect to ground. Because voltage is a relative concept, we could pick *any* point in the circuit and call it ground, and we would still get the same results.

Current is a flow of electrical charge through a path in the circuit. A positive current in a direction is generated by negative charges (electrons) moving in the opposite direction.

We'll start by considering a simple set of components that have two terminals (connection points into the circuit). Each component has a current flowing through it, and a voltage difference across its terminals. Each type of component has some special characteristics that govern the relationship between its voltage and current. We will restrict our attention, in this course, to components that exert a linear constraint on their current and voltage.

One way to model circuits is in terms of their *dynamics*. That is, to think of the currents and voltages in the system and how they change over time. Such systems are appropriately modeled, in fact, using differential or difference equations, connected together into complex systems, as we saw in the last couple of weeks. But for many purposes, the dynamic properties of a circuit converge quickly, and we can directly model the equilibrium state that they will converge to. The combination of the behavior of the components and the structure in which they're connected provides a set of constraints on the equilibrium state of the circuit. We'll work through this view by starting with the constraints that come from the structure, and then examining constraints for three simple types of components.

6.2 Conservation laws

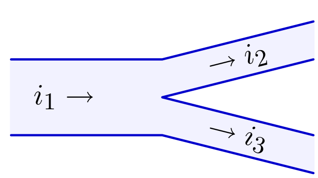
Conservation laws describe properties of a circuit that must hold, no matter what the particular elements are. There are two fundamental conservation laws, one that relates to currents and one that relates to voltages.

6.2.1 Kirchoff's current law

Fluids can be roughly classified as compressible or incompressible. Air is compressible. Doubling the pressure on a volume of air will decrease the volume by a factor of two. By contrast, water is nearly incompressible. Doubling the pressure on a volume of water (from 1 atmosphere to 2) will decrease the volume by less than one part in a billion.

The laws that govern incompressible fluid flows are relatively simple. The net flow of incompressible fluid into a region of fixed volume must be zero. It follows that if there is flow into one part of a fixed volume, then there most be an equal flow out of the other part.

As an example, consider the flow of water through a branching point, as shown below.

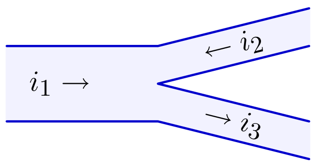


Let i_1 represent the flow into the branching point and i_2 and i_3 represent the flows out of it. Assume that i_1, i_2, and i_3 have the units of volume per unit time (e.g., m^3/s).

Example 1. What is the relation between i_1, i_2, and i_3?

In this example, $i_1 = i_2 + i_3$.

Example 2. What if i_2 represented flow in the opposite direction, as shown below?

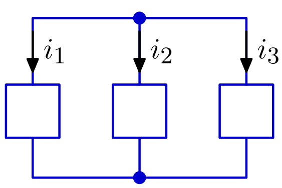


Now $i_1 + i_2 = i_3$ or equivalently $i_1 = -i_2 + i_3$.

Changing the direction of a flow variable is equivalent to changing its sign. For that reason, we think about flow variables as *signed* variables. The direction associated with the variable represents the direction that results when the variable is positive. The flow is in the opposite direction if the variable is negative. Thus, the direction that we associate with a flow is arbitrary. If we happen to associate the one direction, the flow variable will be positive. If we happen to associate the opposite direction, the flow variable will be negative.

The laws for the flow of electrical current are similar to those for the flow of an incompressible fluid. The net flow of electrical current into a **node** must be zero.

The following circuit has three elements, each represented with a box.

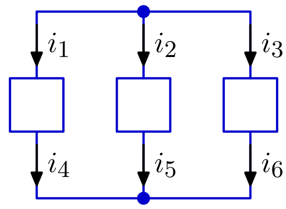


There are two nodes, each indicated by a dot. The net current into or out of each of these nodes is zero. Therefore $i_1 + i_2 + i_3 = 0$.

Similar current relations apply for all nodes in a circuit. Such relations are called **Kirchoff's Current Laws (KCL)** and represent the first of two fundamental conservation laws for circuits.

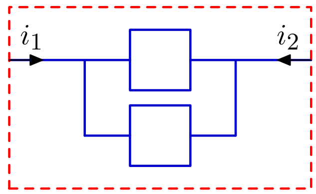
Kirchoff's Current Law: the sum of the currents that flow into a node is zero.

Electrical currents cannot accumulate in elements, so the current that flows into a circuit element must equal to the current that flows out.

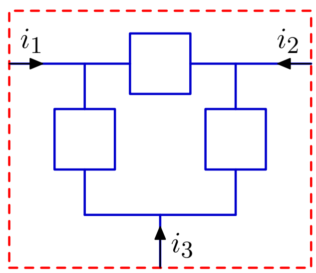


Therefore $i_1 = i_4$, $i_2 = i_5$, and $i_3 = i_6$.

Teach Yourself 1. Part a. A circuit is divided into the part that is inside the dashed red box (below) and the part that is outside the box (not shown). Find an equation that relates the currents i_1 and i_2 that flow **through** the dashed red box.

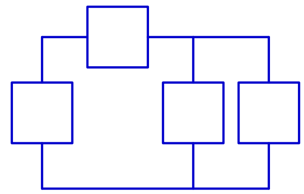


Part b. A more complicated circuit is shown inside the red box below. Find an equation that relates the currents i_1, i_2, and i_3 that flow **through** the dashed red box.



Part c. Generalize your results to an arbitrary closed surface. Explain how to prove the generalization.

Teach Yourself 2. Part a. How many linearly independent[40] KCL equations can be written for the following circuit?

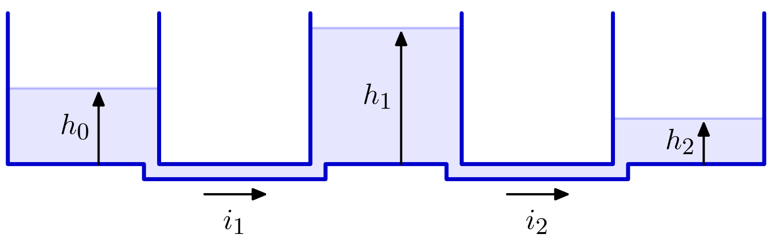


Part b. Generalize your result to determine the number of linearly independent KCL equations in an arbitrary circuit with n nodes. Prove your generalization.

6.2.2 Kirchoff's Voltage Law

What physical mechanisms cause flow? Blood circulates due to pressures generated in the heart. Hydraulic pressure also moves water through the pipes of your house. In similar fashion, voltages propel electrical currents. A voltage can be associated with every node in a circuit; and it is the difference between the voltages at two nodes that excites electrical currents to pass between those nodes.

The flow of water between water storage tanks provides a useful analogy for the flow of electrical currents. Consider three tanks connected as shown below.



[40] None of the equations in a linearly independent set can be written as a linear combination of the other equations. For example,

$$\begin{aligned} x + y &= 1 \\ x + 2y &= 1 \end{aligned}$$

are linearly independent equations. However, the equations

$$\begin{aligned} x + y + z &= 1 \\ x + y &= 1 \\ z &= 0 \end{aligned}$$

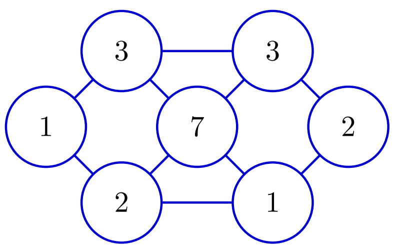
are not, since the last equation is the difference between the first two.

Example 3. What will be the signs of i_1 and i_2?
Water will flow from the middle tank toward the left tank because $h_1 > h_0$. Thus $i_1 < 0$. Similarly, water will flow from the middle tank toward the right tank because $h_1 > h_2$. Thus $i_2 > 0$.

Voltages work similarly for driving electrical currents. It is the difference in voltage between two nodes that drives electrical current between the nodes. Absolute voltage is not important (or even well defined). Voltage differences are all that matter.

Teach Yourself 3. How would the flows in the tank system above change if each height of water in each tank were increased by 1 unit? ... by 1000 units?

Another similarity between the hydraulic pressures in the tank system and voltages in a circuit is that the voltage differences that accumulate along any closed path is zero. This is obvious for an array of tanks. Consider the array illustrated below in a top down view.

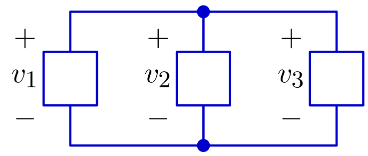


Assume that the lines between circles represent pipes and that the numbers indicate the height (in meters) of the water in each tank. Consider the clockwise path around the perimeter of the network starting and ending with the leftmost tank. The path visits heights of 1, 3, 3, 2, 1, 2, and 1. These heights increase by 2, 0, −1, −1, 1, and −1, which sum to 0.

It's easy to see that the sum of differences in heights around any closed loop will be zero. If it were not, then the height of the first tank would not end up at the same value that it began. This same rule applies to voltage differences, where the rule is called *Kirchoff's Voltage Law (KVL)*.

Kirchoff's Voltage Law: the sum of voltage differences along a closed path is zero.

Example 4. How many KVL equations can be written for the following circuit.



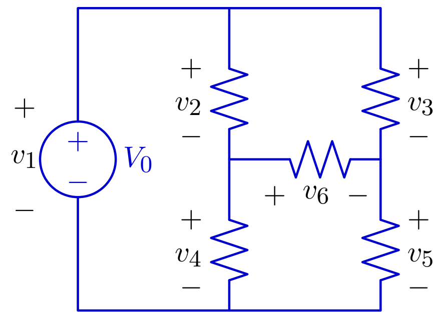
How many of the KVL equations are linearly independent?

There are three loops through this circuit. One through the left two elements yields the KVL equation $-v_1 + v_2 = 0$ or $v_1 = v_2$. One through the right two elements yields the KVL equation $-v_2 + v_3 = 0$ or $v_2 = v_3$. One through the outer two elements yields the KVL equation $-v_1 + v_3 = 0$ or $v_1 = v_3$.

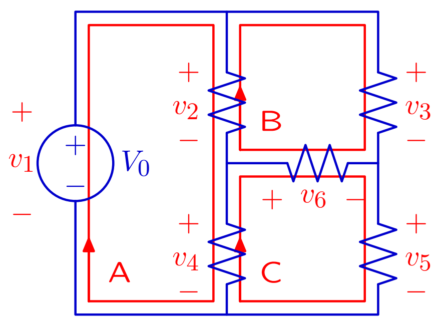
Only two of the three KVL equations are linearly independent, since each equation can be derived from the other two.

The particularly simple solution to this circuit is that $v_1 = v_2 = v_3$. This solution could also have been derived directly from the fact that there are just two nodes, and therefore only one possible potential difference.

Example 5. How many different KVL equations can be written for this circuit?



How many of the KCL equations are linearly independent?
One KVL equation can be written for every possible closed loop through the circuit. The most obvious loops are A, B, and C shown below.

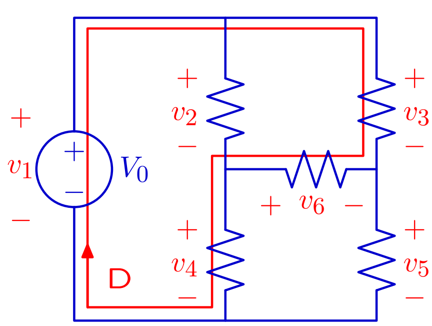


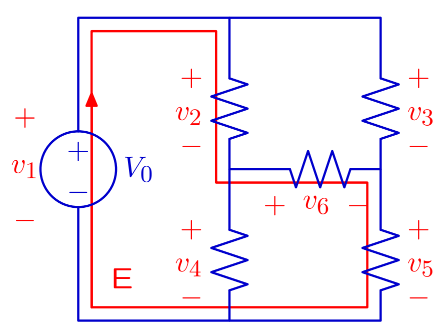
A: $-v_1 + v_2 + v_4 = 0$

B: $-v_2 + v_3 - v_6 = 0$

C: $-v_4 + v_6 + v_5 = 0$

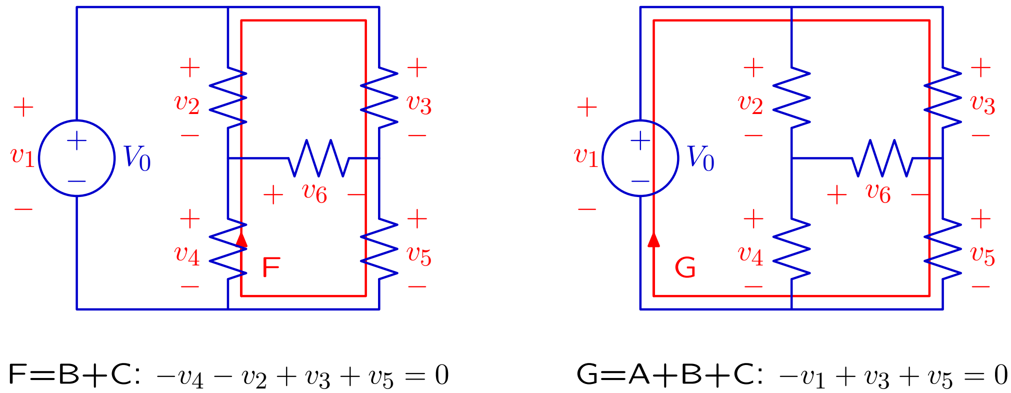
But there are several more, shown below.





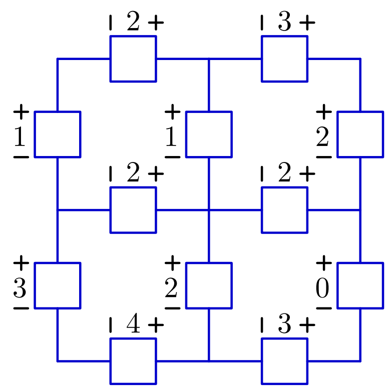
D=A+B: $-v_1 + v_3 - v_6 + v_4 = 0$

E=A+C: $-v_1 + v_2 + v_6 + v_5 = 0$



These seven equations are not linearly independent. Equation D is the sum of Equations A and B (notice that the v_2 terms in Equations A and B cancel). Similarly, equation E is the sum of Equations A and C, equation F is the sum of Equations B and C, and equation G is the sum of Equations A, B, and C. Thus this circuit has three independent KVL equations and four additional dependent KVL equations.

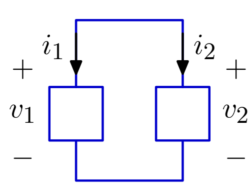
Teach Yourself 4. The following set of voltages are not consistent with Kirchoff's voltage laws but can be made consistent by changing just one. Which one?



Prove that the answer is unique.

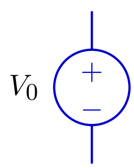
6.3 Circuit elements

Kirchoff's Current and Voltage Laws do not alone determine the behavior of a circuit. Consider the following circuit with just two elements. There are two elements, so there are two element currents i_1 and i_2 as well as two element voltages v_1 and v_2, all of which are unknown.

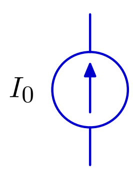


Example 6. List all of the KCL and KVL equations for this circuit.
There are two nodes in this circuit. As we saw in **TeachYourself 2**, this implies that there is a single independent KCL equation, which is $i_1 + i_2 = 0$. There is a single loop and therefore a single KVL equation: $v_1 - v_2 = 0$. Thus we have four unknowns and just two equations. More information is needed to solve for the currents and voltages.

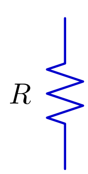
In general, every electrical element imposes a relation between the voltage across the element and the current through the element. This relation is called a **constitutive** or **element** relation. The simplest relations are those for **sources**. A voltage source is an element whose voltage is a constant (e.g., $v = V_0$) independent of the current through the element. We will denote a voltage source by a circle enclosing plus and minus symbols to denote voltage. A number beside the source will indicate the constant voltage generated by the source, as illustrated below. Voltages are specified with units called **volts** which are abbreviated as V.



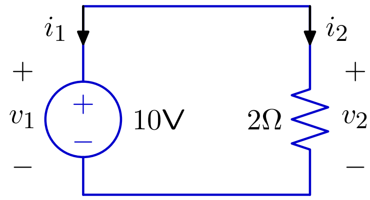
A current source is an element whose current is a constant (e.g., $i = I_0$) independent of the voltage across the element. We will denote a current source by a circle enclosing an arrow to denote current. A number beside the source will indicate the constant current generated by the source, as illustrated below. Currents are specified with units called **amperes** or simply **amps** which are abbreviated as A.



Our third simple element is a **resistor**, in which the voltage is proportional to the current and the proportionality constant is the resistance R, so that $v_R = Ri_R$. We will denote a resistor as follows. Resistances have the dimensions of **ohms** (abbreviated as Ω), which are equivalent to volts divided by amps.

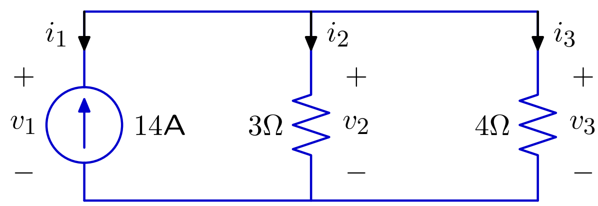


A circuit consisting of a 10V voltage source and 2Ω resistor can easily be solved.



Example 7. Find v_1, v_2, i_1, and i_2.
As before, $i_1 + i_2 = 0$ and $v_1 = v_2$. However, now we also know two consitutive relations: $v_1 = 10\text{V}$ and $v_2 = 2\Omega \times i_2$. Solving, we find that $v_1 = v_2 = 10\text{V}$, $i_2 = v_2/2\Omega = 5\text{A}$, and $i_1 = -i_2 = -5\text{A}$.

Example 8. Determine the currents and voltages in the following circuit.

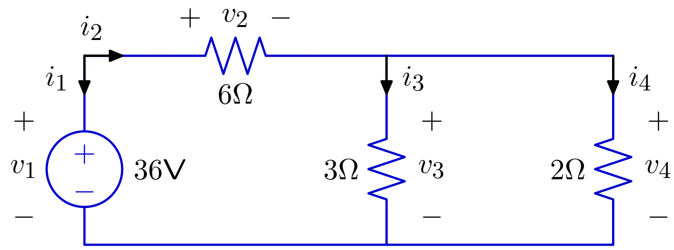


There is a single independent KCL equation: $i_1 + i_2 + i_3 = 0$. There are two independent KVL equations that together yield $v_1 = v_2 = v_3$. Thus, KCL and KVL together yield three equations in six unknowns. We need three more equations — one from each element. From the current source constitutive relation we have $i_1 = -14\text{A}$. From the resistor constitutive relations we have $v_2 = 3\Omega \times i_2$ and $v_3 = 4\Omega \times i_3$. Solving

$$v_1 = v_2 = 3\Omega \times i_2 = v_3 = 4\Omega \times i_3 = 4\Omega \times (-i_1 - i_2) = 4\Omega \times (14 - i_2)$$

so that $3\Omega \times i_2 = 4\Omega \times (14 - i_2)$ so that $i_2 = 8\text{A}$. Therefore $i_1 = -14\text{A}$, $i_2 = 8\text{A}$, and $i_3 = 6\text{A}$. Also, $v_1 = v_2 = 3\Omega \times 8\text{A} = v_3 = 4\Omega \times 6\text{A} = 24\text{V}$.

Example 9. Solve the following circuit.



Answer:

$$i_1 = -5\text{A}\,; \quad i_2 = 5\text{A}\,; \quad i_3 = 2\text{A}\,; \quad i_4 = 3\text{A}\,;$$

$$v_1 = 36\text{V}\,; \quad v_2 = 30\text{V}\,; \quad v_3 = 6\text{V}\,; \quad v_4 = 6\text{V}$$

6.4 Solving circuits

Given a description of the circuit's wiring and a specification of the individual components, we can *solve* the circuit to answer questions about particular voltages or currents in the circuit. There are several ways to approach this problem: we will start with a completely systematic method that is particularly useful if we want to solve circuits using a computer; then, we'll see some simpler method and patterns that are easier for humans to use.

The most obvious strategy would be to write down all the KVL and KCL equations, add the constituent equations for each element, and then solve for the voltage and current at each component. But we have a problem with redundant KVL equations. The voltages across circuit elements are constrained by KVL so that the sum of the voltages around any closed loop is zero. However, there are many possible closed loops through even simple circuits (see **section 6.2.2**). If we are dealing with a planar circuit,[41] then we know that all loops can be written as combinations of primitive loops with non-overlapping interiors (see **example 5**).

Teach Yourself 5. Each of four nodes are connected to each of the others by a 1Ω resistor. How many resistors are there? Is the network planar?

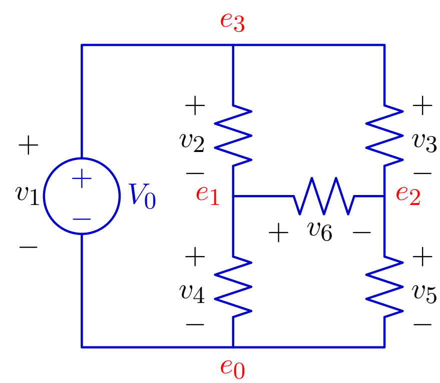
Teach Yourself 6. Each of five nodes are connected to each of the others by a 1Ω resistor. How many resistors are there? Is the network planar?

So far, the only general way that we have seen to deal with this redundancy is to (1) find **all** of the possible KVL equations, and then (2) eliminate the redundant ones. This process seems needlessly complicated.

[41] A circuit is "planar" if it is possible to draw all of the elements and all of the lines that connect the elements without any of the lines or elements crossing over one another.

One good way to eliminate KVL redundancy is to use **node voltages**. Node voltages are voltages associated with circuit nodes rather than with circuit elements. Given a complete set of node voltages, it is easy to determine the element voltages: each element voltage is the difference between the voltage of the node at the positive end of the element minus that at its negative end.

Example 10. Determine expressions for each of the element voltages from the node voltages e_0, e_1, e_2, and e_3.



Answers: $v_1 = e_3 - e_0$; $v_2 = e_3 - e_1$; $v_3 = e_3 - e_2$; $v_4 = e_1 - e_0$; $v_5 = e_2 - e_0$; $v_6 = e_1 - e_2$;

Example 11. How would the element voltages in the previous example change if 1V were added to each of the node voltages?

Adding a constant voltage to each of the node voltages of a circuit has no effect on any of the element voltages. Thus, substituting node voltages for element voltages removes redundancies that are implicit in KCL equations, but also introduces a new redundancy, which we can think of as a global offset voltage. If we are only interested in element voltages (which is typically the case), we can exclude effects of a global offset voltage by arbitrarily assigning the potential of one node to zero. We refer to that special node with zero potential as the “ground.”

We will explore two methods that use node voltages: the node-voltage-and-component-current (NVCC) method, which has variables for voltages at all of the nodes and currents through all of the components, and the node method, which only has variables for voltages at the nodes. The NVCC method is completely general, but rather long-winded (and is perhaps better for computers to use than humans). The node method is terser, with potentially many fewer equations, but can become complicated when there are multiple voltage sources.

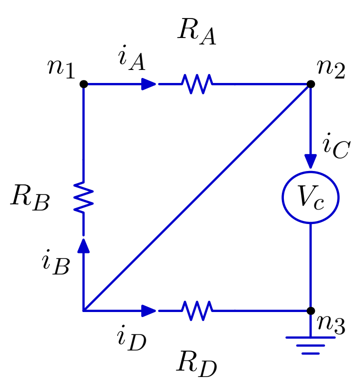
6.4.1 NVCC Method

In the NVCC method, we take the following steps:

1. Label all of the *nodes* (places where there is a wire between two or more components) with names $n_1, \ldots, n_n$, and make variables $v_1, \ldots, v_n$, one for the voltage at each node.
2. Declare one of them, n_g, to be the *ground node* (this can be any node; the voltages will be computed relative to the voltage at that node) and set $v_g = 0$.
3. Make current variables $i_1, \ldots, i_m$ for each component (resistor or source) in the network. Label a direction for each current in your network (it doesn't matter what directions you pick as long as you handle them consistently from here on out).
4. Write down $n-1$ KCL equations, one for each node except for n_g. These equations assert that the sum of currents entering each node is 0.
5. Write down m constitutive equations, one for each component, describing that component's linear relationship between its current i_k and the voltage difference across the component. The voltage across the component is $v_{k+} - v_{k-}$, where v_{k+} is the node voltage at the positive terminal of the component and v_{k-} is the node voltage at its negative terminal voltage. The direction of the current defines what constitutes the 'positive' and 'negative' terminals of the component: the current runs from positive to negative.

   For a resistor with resistance R, the equation is $v_{k+} - v_{k-} = i_k R$; for a voltage source with voltage V_s, the equation is $v_{k+} - v_{k-} = V_s$; for a current source with current C_s, the equation is $i_k = C_s$.
6. Solve these equations to determine the node voltage and component currents.

So, for this circuit,



using the current directions for each component as drawn, we have the following equations:

- **Ground**

$$v_3 = 0 \ .$$

- **KCL**

$$i_A - i_B - i_D - i_C = 0$$
$$i_B - i_A = 0 \ .$$

- **Constitutive equations**

$$
\begin{aligned}
(v_1 - v_2) &= i_A \cdot R_A \\
(v_2 - v_1) &= i_B \cdot R_B \\
(v_2 - v_3) &= i_D \cdot R_D \\
v_2 - v_3 &= V_c \quad .
\end{aligned}
$$

So, now, if we know R_A, R_B, R_D, and V_C, which are the specifications of our components, we have 7 linear equations in 7 unknowns (v_1, v_2, v_3, i_A, i_B, i_C, and i_D). Just a small (though possibly tedious) matter of algebra, and we're done.

As an example, let $R_A = 100\Omega$, $R_B = 200\Omega$, $R_D = 100\Omega$, and $V_C = 10V$. Then, we get $v_2 = 10V$; $i_A = i_B = 0A$ (that's reasonable: why would any current bother going that way, when it can just run through the diagonal wire?); and $i_D = 0.1A$, which is pretty straightforward.

6.4.2 Solution strategy

You can see that the KCL equations are all in terms of currents; and the constitutive equations give us direct expressions for almost all of the currents. So, a good solution strategy is to work primarily with the KCL equations, trying to eliminate as many current variables as possible, by substituting in expressions derived from the constitutive equations. In the example above, here are the KCL equations:

$$
\begin{aligned}
i_A - i_B - i_D - i_C &= 0 \\
i_B - i_A &= 0 \quad .
\end{aligned}
$$

Now, substituting in expressions from the constitutive equations, we can rewrite them as:

$$
\begin{aligned}
\frac{v_1 - v_2}{R_A} - \frac{v_2 - v_1}{R_B} - \frac{V_c}{R_D} - i_C &= 0 \\
\frac{v_2 - v_1}{R_B} - \frac{v_1 - v_2}{R_A} &= 0 \quad .
\end{aligned}
$$

We also know, from the ground equation $v_3 = 0$, that $v_2 = V_C$, so we can rewrite our equations as:

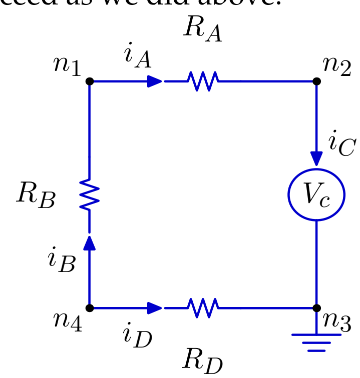
$$
\begin{aligned}
\frac{v_1 - V_c}{R_A} - \frac{V_c - v_1}{R_B} - \frac{V_c}{R_D} - i_C &= 0 \\
\frac{V_c - v_1}{R_B} - \frac{v_1 - V_c}{R_A} &= 0 \quad .
\end{aligned}
$$

> In most cases, you can write equations of this form down directly, without bothering to write the KCL and the constituent equations separately, and then substituting in for the currents.

Now we only have two unknowns: v_1 and i_C, and two equations. The second equation tells us that $V_c = V_1$, and, substituting that into the first equation, we have

$$
i_C = \frac{V_c}{R_D}
$$

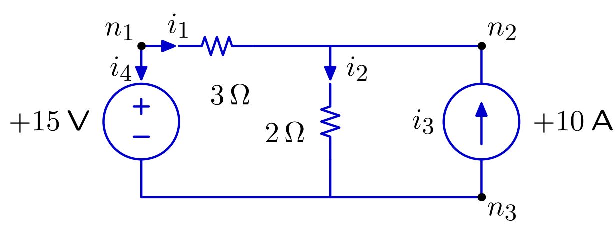
Exercise 6.1. What happens when we take out the diagonal wire? We have to add a new node, v_4, and then proceed as we did above.



Ultimately, we can find that $i_A = i_B = i_C = -0.025A$, $i_D = 0.025A$, $v_1 = 7.5V$, $v_2 = 10V$, and $v_4 = 2.5V$. Verify this solution by writing out the equations and solving them using a process similar to the one outlined above.

6.4.3 An example

Let's try to find the node voltages and constituent currents for this circuit:



NVCC

Here are the steps of the NVCC method:

1. We have labeled the nodes n_1, n_2, and n_3. Remember that we don't need a node for every corner or join in the circuit: a single 'node' can cover a stretch of wire with several connections, because it will all have the same voltage as long as no components intervene.
2. Declare one of them to be the *ground node*. When there's a single voltage source, it is conventional (but not necessary!) to set its negative terminal to be the ground node; so we'll set $v_3 = 0$.
3. Make current variables for each component in the network and label them with arrows. We did this already in the figure.
4. Write down $n - 1$ KCL equations, one for each node except for the ground node.

$$-i_4 - i_1 = 0$$

$$i_1 - i_2 + i_3 = 0$$

5. Write down m constitutive equations, one for each component, describing that component's linear relationship between its current i_k and the voltage difference across the component.

$$\begin{aligned} v_1 &= 15 \\ v_1 - v_2 &= 3i_1 \\ v_2 - v_3 &= 2i_2 \\ i_3 &= 10 \end{aligned}$$

6. Solve these equations to determine the node voltage and component currents. There are lots of ways to work through this algebra. Here's one meandering path to the right answer.

We can do some plugging in of v_1, v_3, and i_3 to simplify matters.

$$\begin{aligned} 15 - v_2 &= 3i_1 \\ v_2 &= 2i_2 \\ -i_4 - i_1 &= 0 \\ i_1 - i_2 + 10 &= 0 \end{aligned}$$

Plugging expressions for i_1 and i_2 derived from the first two equations into the last one, we get

$$\frac{15 - v_2}{3} - \frac{v_2}{2} = -10$$

so

$$v_2 = 18$$

And now it's easy to see that $i_1 = -1$, $i_2 = 9$, and $i_4 = 1$.

NVCC Shortcut

If we want to shortcut the NVCC method, we can follow the first three steps above, but then try to go straight to versions of the KCL equations that substitute in expressions for current that are derived from the constituent constraints. So, we could observe that if $v_3 = 0$ then $v_1 = 15$. And then we could write the KCL equations as:

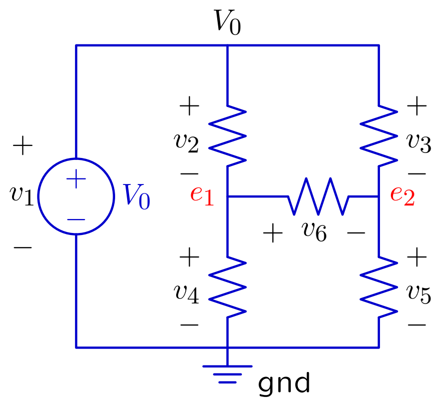
$$\begin{aligned} -i_4 - \frac{15 - v_2}{3} &= 0 \\ \frac{15 - v_2}{3} + 10 - \frac{v_2}{2} &= 0 \end{aligned}$$

Solving the second equation we find that $v_2 = 18$; from this we can easily determine the other relevant quantities.

6.4.4 Node method (optional)

Node analysis is similar to NVCC, but with the goal of writing down one equation per node.

We can further simplify the node representation of circuits that contain voltage sources. When a voltage source connects two nodes, it constrains the two node voltages so that one can be determined from the other, as shown in the following figure, where e_0 in the previous figure is taken as ground.

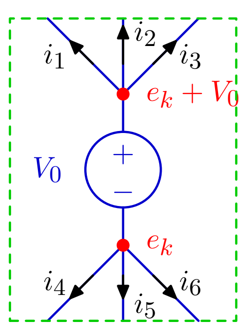


So far, we have seen that node voltages are a concise representation of all of the information that is necessary to compute the element voltages of a circuit. Furthermore, node voltages automatically guarantee that KVL is satisfied by all of the element voltages. To complete the analysis of a circuit using node voltages, we must next consider the element currents and KCL.

In general, one can write KCL at each of the nodes of a circuit. However, as we saw in **TeachYourself 2**, one of those equations will always be linearly dependent on the others. Thus, if the circuit has n nodes, we can write $n-1$ independent KCL equations.

If there are no voltage sources in the circuit, then we can express each KCL equation as a sum of element currents that can each be determined from the node voltages by using the element's constitutive equation. The result is a system of $n-1$ KCL equations that can be expressed in terms of $n-1$ node voltage variables.

If there are voltage sources in the circuit, then two additional issues arise. First, each voltage source connects two nodes, but is represented by just one node variable (since the voltage of the source determines the voltage at the other node). Second, the current through a voltage source **cannot** be determined from its constitutive equation. The constitutive equation for a voltage source is $v_s = V_0$, which must be true for all currents. Thus, knowing that $v_s = V_0$ provides no information about the current through the source. These constraints are illustrated below.



A consistent set of equations can be derived by treating the entire voltage source as a single node, represented by the dashed green box above. The KCL equation for that node,

$$i_1 + i_2 + i_3 + i_4 + i_5 + i_6 = 0$$

involves element currents that can each be written (using their constitutive equations) in terms of node voltages outside the box plus e_k and $e_k + V_0$. Like the KCL equations outside the box, this KCL equation adds a single new unknown, which is e_k.

Example 12. Does the above analysis imply that the current through the voltage source is zero?

No. Let i_v represent the current through the voltage source. KCL at the top node requires that

$$i_1 + i_2 + i_3 + i_v = 0$$

KCL at the bottom node requires that

$$i_4 + i_5 + i_6 - i_v = 0$$

The sum of these equations is the KCL equation given above for the dashed green box. It holds reguardless of the value of the current through the voltage source. To find the value of the current through the current source, first solve for i_1, i_2, i_3, i_4, i_5, i_6, and e_k, ignoring i_v. Then $i_v = -(i_1 + i_2 + i_3)$, which will be the same as $i_4 + i_5 + i_6$.

Teach Yourself 7. How would you handle a circuit in which i_1 (above) connects directly to another voltage source outside the dashed green box?

Node Analysis: Summary

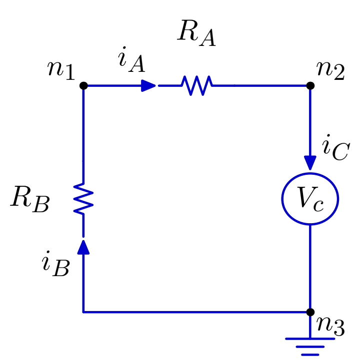
- Assign a voltage expression to each node in the circuit, as follows:
  - Assign a voltage of zero to one node, which is called the "ground."
  - Assign voltages to nodes that are connected to voltage sources so that the voltage source relations are satisfied:
    - ⋆ If the voltage on one terminal of a voltage source is known (e.g., ground), then use the voltage source relation to determine the node voltage on the other terminal.
    - ⋆ If neither terminal of the voltage source connects to a known voltage, then assign a variable to represent the node voltage at one terminal and use the voltage source relation to determine the node voltage at the other terminal.
  - Assign variables to represent the voltages at all remaining nodes.
- For each element in the circuit that is not a voltage source, use the associated constitutive law to express the current through the element in terms of the node voltages at its terminals.
- Write one KCL equation for each unknown node voltage, as follows:
  - Express KCL for each node that is not connected to a voltage source as a sum of element currents.
  - For each voltage source, write a single KCL to express the sum of currents entering the nodes connected to both its positive **and** negative terminals.

6.4.5 Common Patterns

There are some common patterns of resistors that are important to understand and that can be used over and over again as design elements. In this section, we will lead into a systematic approach to solving circuits. The first step in that approach is to associate voltages with nodes (points at which components connect) in the circuit rather than with the components themselves. This only makes sense if we establish a fixed reference voltage. We pick a particular node in the circuit (it won't matter which one, but it is, for example, conventional to pick the negative terminal of a voltage source) and call it *ground* and assert that it has voltage 0. Now, when we speak of a node in the circuit having voltage v, what we mean is that the voltage difference between that node and the ground node is v. Ground is indicated in a circuit diagram by a special single-terminal component made of three parallel lines.

6.4.5.1 Resistors in series

The figure below shows two resistors connected together in a circuit with a voltage source, the negative terminal of which is connected to ground.



It induces a simple set of constraints (remember that we don't need a KVL equation for the ground node):

$$i_A - i_C = 0$$
$$i_B - i_A = 0$$
$$v_3 = 0$$
$$v_1 - v_2 = i_A \cdot R_A$$
$$v_3 - v_1 = i_B \cdot R_B$$
$$v_2 - v_3 = V_c$$

What happens when we solve? First, it's easy to see that because there's a single loop, KCL implies that the current across each of the nodes is the same. Let's call it i. Now, we can add together the fourth and fifth equations, and then use the last equation to get

$$v_3 - v_2 = i_A R_A + i_B R_B$$
$$v_3 - v_2 = i(R_A + R_B)$$
$$-V_c = i(R_A + R_B)$$
$$-i = \frac{V_c}{R_A + R_B}$$

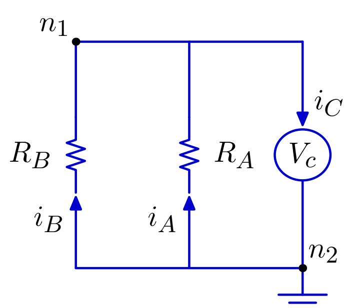
The interesting thing to see here is that we get exactly the same result as we would have had if there were a single resistor R, with resistance $R_A + R_B$. So, if you ever see two or more resistors in series in a circuit, with no other connections from the point between them to other components, you can treat them as if it were one resistor with the sum of the resistance values. This is a nice small piece of abstraction.

It might bother you that we got something that looks like $v = -iR$ instead of $v = iR$. Did we do something wrong? Not really. The reason that it seems funny is that the directions we picked for the currents i_A and i_B turn out to be "backwards", in the sense that, in fact, the current is running in the other direction, given the way we hooked them up to the voltage source. But the answer is still correct.

Exercise 6.2. Consider the circuit we just analayzed. You should be able to construct an equivalent circuit with only one resistor. What is its resistance value?

6.4.5.2 Resistors in parallel

Now, in this figure,



we have a simple circuit with two resistors in parallel. Even though there are a lot of wires being connected together, there are really only two *nodes*: places where multiple components are connected. Let's write down the equations governing this system.

First, applying KCL to n_1,

$$i_A + i_B - i_C = 0 \ .$$

Now, setting v_2 to ground, and describing the individual components, we have:

$$v_2 = 0$$
$$v_2 - v_1 = i_A \cdot R_A$$
$$v_2 - v_1 = i_B \cdot R_B$$
$$v_1 - v_2 = V_c$$

We can simplify this last set of constraints to

$$-V_C = i_A \cdot R_A$$
$$-V_C = i_B \cdot R_B$$

so

$$i_A = -\frac{V_c}{R_A}$$
$$i_B = -\frac{V_c}{R_B}$$

Plugging these into the KCL equation, we get:

$$i_A + i_B - i_C = 0$$
$$-\frac{V_c}{R_A} - \frac{V_c}{R_B} = i_C$$
$$-V_c \frac{R_A + R_B}{R_A R_B} = i_C$$
$$-V_c = i_C \frac{R_A R_B}{R_A + R_B}$$

What we can see from this is that two resistances, R_A and R_B, wired up in parallel, act like a single resistor with resistance

$$\frac{R_A R_B}{R_A + R_B} \quad .$$

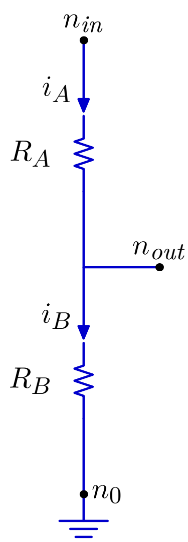
This is another common pattern for both analysis and design. If you see a circuit with parallel resistors connected at nodes n_1 and n_2, you can simplify it to a circuit that replaces those two paths between n_1 and n_2 with a single path with a single resistor.

To get some intuition, think about what happens when R_A is much bigger than R_B. Because the voltage across both resistors is the same, it must be that the current through R_A is much smaller than the current through R_B. The appropriate intuition here is that current goes through the resistor that's easier to pass through.

Exercise 6.3. If $R_A = 10000\Omega$ and $R_B = 10\Omega$, what is the effective resistance of R_A and R_B in parallel?

6.4.5.3 Voltage divider

The figure below shows part of a circuit, in a configuration known as a *voltage divider*.



Using what we know about circuit constraints, we can determine the following relationship between V_{out} and V_{in}:

$$V_{out} = \frac{R_B}{R_A + R_B} V_{in} \quad .$$

Let's go step by step. Here are the basic equations:

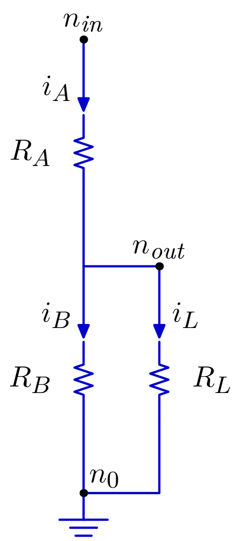
$$\begin{aligned} v_0 &= 0 \\ i_A - i_B &= 0 \\ V_{in} - V_{out} &= i_A R_A \\ V_{out} - v_0 &= i_B R_B \end{aligned}$$

We can start by seeing that $i_A = i_B$; let's just call it i. Now, we add the last two equations to each other, and do some algebra:

$$\begin{aligned} V_{in} - v_0 &= iR_A + iR_B \\ V_{in} &= i(R_A + R_B) \\ i &= \frac{V_{in}}{R_A + R_B} \\ V_{in} - V_{out} &= iR_A \\ V_{in} - V_{out} &= V_{in}\frac{R_A}{R_A + R_B} \\ V_{in}(R_A + R_B) - V_{out}(R_A + R_B) &= V_{in}R_A \\ V_{in}R_B &= V_{out}(R_A + R_B) \\ V_{out} &= V_{in}\frac{R_B}{R_A + R_B} \end{aligned}$$

So, for example, if $R_A = R_B$, then $V_{out} = V_{in}/2$. This is a very handy thing: if you need a voltage in your circuit that is between two values that you already have available, you can choose an appropriate R_A and R_B to create that voltage.

Well, almost. When we wrote $i_A - i_B = 0$, we were assuming that there was no current flowing out V_{out}. But, of course, in general, that won't be true. Consider the following figure:



We've shown an additional "load" on the circuit at V_{out} with a resistor R_L standing for whatever resistance that additional load might offer to the ground node n_0. This changes matters considerably.

To see what is going to happen, we could solve the whole circuit again. Or, we could observe that, between the node labeled V_{out} and n_0, we have two resistances, R_B and R_L in parallel. And we've already see that resistances in parallel behave as if they are a single resistor with value $R_B R_L/(R_B + R_L)$. So, (you do the algebra), our result will be that

$$V_{out} = V_{in} \frac{R_B}{R_A + R_B + \frac{R_A R_B}{R_L}} .$$

The lesson here is that the modularity in circuits is not as strong as that in programs or our difference equation models of linear systems. How a circuit will behave can be highly dependent on how it is connected to other components. Still, the constraints that it exerts on the overall system remain the same.

6.5 Circuit Equivalents

We just saw that pieces of circuits cannot be abstracted as input/output elements; the actual voltages and currents in them will depend on how they are connected to the rest of a larger circuit. However, we can still abstract them as sets of constraints on the values involved.

In fact, when a circuit includes only resistors, current sources, and voltage sources, we can derive a much simpler circuit that induces the same constraints on currents and voltages as the original one. This is a kind of abstraction that's similar to the abstraction that we saw in linear systems: we can take a complex circuit and treat it as if it were a much simpler circuit.

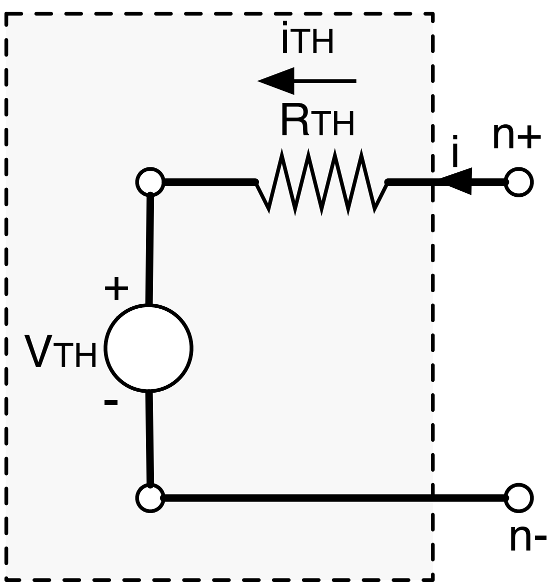
If somebody gave you a circuit and put it in a black box with two wires coming out, labeled + and -, what could you do with it? You could try to figure out what constraints that box puts on the voltage between and current through the wires coming out of the box.

We can start by figuring out the *open-circuit voltage* across the two terminals. That is the voltage drop we'd see across the two wires if nothing were connected to them. We'll call that V_{oc}. Another thing we could do is connect the two wires together, and see how much current runs through them; this is called the *short-circuit current*. We'll call that i_{sc}.

Because all of the relationships in our circuits are linear, these two values are sufficient to characterize the constraint that this whole box will exert on a circuit connected to it. The constraint will be a relationship between the voltage across its terminals and the current flowing through the box. We can derive it by using Thévenin's theorem:

> Any combination of current sources, voltage sources, and resistances with two terminals can be replaced by a single voltage source V_{th} and a single series resistor R_{th}. The value of V_{th} is the open circuit voltage at the terminals V_{oc}, and the value of R_{th} is V_{th} divided by the current with the terminals short circuited ($-i_{sc}$).

Let's look at a picture, then an example.



Here, we show a picture of a black (well, gray) box, abstracted as being made up of a circuit with a single voltage source V_{th} and a single resistor R_{th} in series. The open-circuit voltage from n_+ to n_- is clearly V_{th}. The short-circuit current i_{sc} (in the direction of the arrow) is $-V_{th}/R_{th}$. So, this circuit would have the desired measured properties.[42]

Now, here is an actual circuit. We'll compute its associated open-circuit voltage and short-circuit current, construct the associated *Thévenin equivalent* circuit, and be sure it has the same properties.

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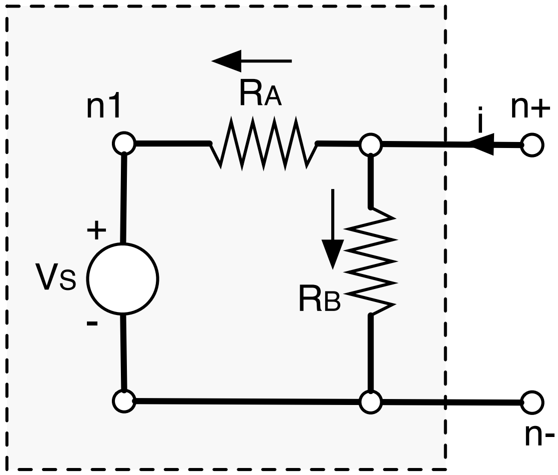
[42] The minus sign here can be kind of confusing. The issue is this: when we are treating this circuit as a black box with terminals n_+ and n_-, we think of the current flowing *out* of n_+ and *in* to n_-, which is consistent with the voltage difference $V_{th} = V_+ - V_-$. But when we compute the short-circuit current by wiring n_+ and n_- together, we are continuing to think of i_{sc} as flowing out of n_+, but now it is coming *out* of n_- and *in* to n_+, which is the opposite direction. So, we have to change its sign to compute R_{th}.



The first step is to compute the open-circuit voltage. This just means figuring out the difference between the voltage at nodes n_+ and n_-, under the assumption that the current $i = 0$. An easy way to do this is to set n_- as ground and then find the node voltage at n_+. Let's write down the equations:

$$\begin{aligned}
v_+ - v_1 &= i_A R_A \\
v_1 - v_- &= V_s \\
v_+ - v_- &= i_B R_B \\
-i_A - i_B &= 0 \\
i_A - i_S &= 0 \\
v_- &= 0
\end{aligned}$$

We can solve these pretty straightforwardly to find that

$$v_+ = V_s \frac{R_B}{R_A + R_B} \quad .$$

So, we know that, for this circuit, $V_{th} = V_s \frac{R_B}{R_A+R_B}$.

Now, we need the short-circuit current, i_{sc}. To find this, imagine a wire connecting n_+ to n_-; we want to solve for the current passing through this wire. We can use the equations we had before, but adding equation 6.1 wiring n_+ to n_-, and adding the current i_{sc} to the KCL equation 6.2.

$$\begin{aligned}
v_+ - v_1 &= i_A R_A \\
v_1 - v_- &= V_s \\
v_+ - v_- &= i_B R_B \\
v_+ &= v_- && (6.1) \\
i_{sc} - i_A - i_B &= 0 && (6.2) \\
i_A - i_S &= 0 \\
v_- &= 0
\end{aligned}$$

We can solve this system to find that

$$i_{sc} = -\frac{V_s}{R_A} \quad ,$$

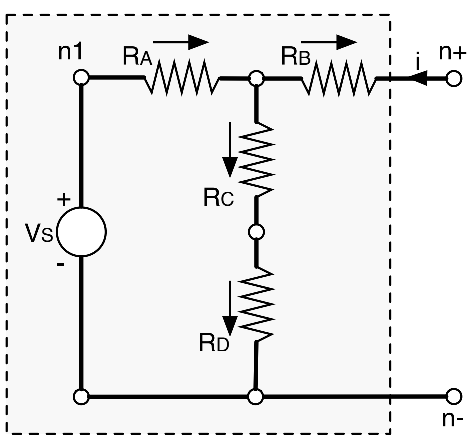
and therefore that

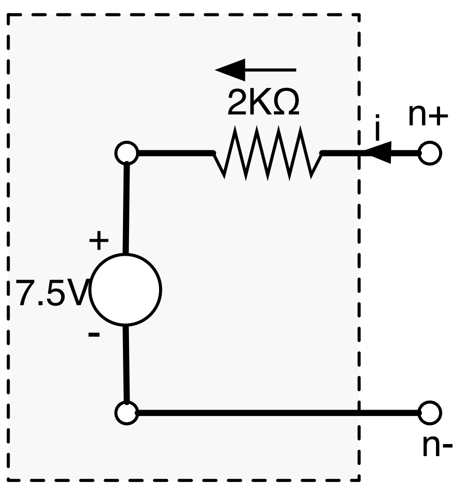
$$\begin{aligned} R_{th} &= -\frac{V_{th}}{i_{sc}} \\ &= V_s \frac{R_B}{R_A + R_B} \frac{V_s}{R_A} \\ &= \frac{R_A R_B}{R_A + R_B} \end{aligned}$$

What can we do with this information? We could use it during circuit analysis to simplify parts of a circuit model, individually, making it easier to solve the whole system. We could also use it in design, to construct a simpler implementation of a more complex network design. One important point is that the Thévenin equivalent circuit is not exactly the same as the original one. It will exert the same constraints on the voltages and currents of a circuit that it is connected to, but will, for example, have different heat dissipation properties.

6.5.1 Example

Here's another example.

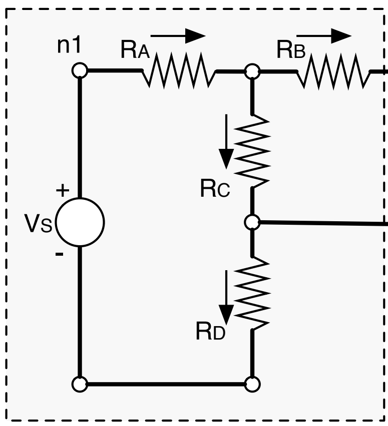




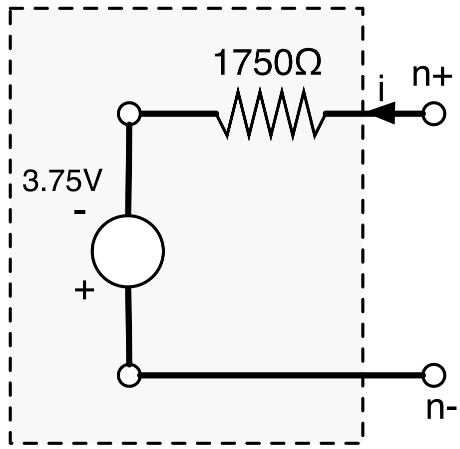
Another example circuit Its Thévenin equivalent

It's a bit more hassle than the previous one, but you can write down the equations to describe the constituents and KCL constraints, as before. If we let $R_A = 2K\Omega$, $R_B = R_C = R_D = 1K\Omega$, and $V_S = 15V$, then we can solve for $V_{th} = 7.5V$ and $R_{th} = 2K\Omega$. So, it is indistinguishable by current and voltage from the circuit shown in the equivalent circuit.

Here is the same circuit, but with the connections that run outside the box made to different nodes in the circuit.



Same circuit, different ports



Its Thévenin equivalent

Note also that the top lead is marked n_- and the bottom one n_+. If we solve, using the same values for the resistors and voltage source as before, we find that $V_{th} = -3.75V$ and $R_{th} = 1750\Omega$. We show the Thévenin equivalent circuit next to the original. We've changed the polarity of the voltage source and made it 3.75V (instead of having the + terminal at the top and a voltage of -3.75), but that's just a matter of drawing.

These results are quite different: so, the moral is, it matters which wires you connect up to what!

6.5.2 Norton Equivalents

It is also possible to derive an equivalent circuit consisting of a single resistor in parallel with a single current source. Such a circuit is called a *Norton equivalent* circuit. The process for deriving Norton equivalents is similar in spirit to the process outlined above.

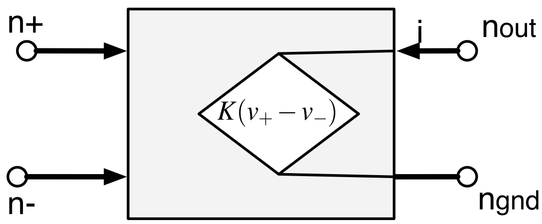
6.6 Op Amps

So far, we have considered circuits with resistors, voltage sources, and current sources. Now we are going introduce a new component, called an *operational amplifier* or op-amp, for short. Op-amps are a crucial tool for decoupling some aspects of complex circuits, so that they can be designed independently and then connected. They are also important for amplifying small voltage differences into larger ones, and can be used to scale, invert, and add voltages.

6.6.1 Voltage-controlled voltage sources

In fact, op-amps are complex assemblages of transistors and other circuit components. We'll need a simpler model than that of what they do, so that we can analyze circuits that contain them, and design new circuits that use them.

We will think about an op-amp as a *voltage-controlled voltage source* (VCVS). Op-amps have four terminals, two of which are often called the input terminals and two, the output terminals. We can think of the op-amp as a voltage source, in which the voltage difference across the output terminals is controlled by the voltage difference between the input terminals. The crucial thing about a voltage source is that it maintains the voltage difference that it is trying to source, no matter what it is connected to.



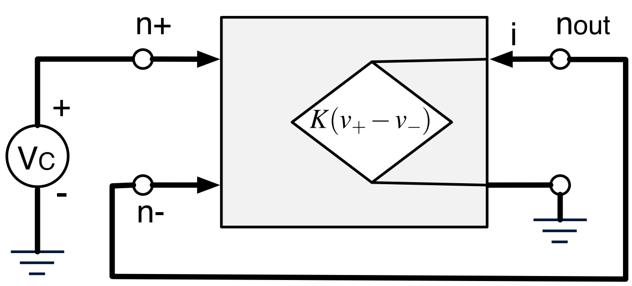
A model of the constraint that an op-amp exerts on the circuit it is connected to relates the voltages at its four terminals in this way:

$$v_{out} - v_{gnd} = K(v_+ - v_-) \quad ,$$

where K is a very large gain, on the order of 10,000, and asserts that no current flows into the op amp.

$$i_+ = i_- = 0 \quad .$$

So, we can think of n_{out} and n_{gnd} as constituting a voltage source, whose voltage is defined to be $K(v_+ - v_-)$. We can see it as *amplifying* the voltage difference $v_+ - v_-$. It's hard to really understand how this model works without seeing it in context. So, we'll study an op-amp that is wired up in a configuration called a *voltage follower*:



Note that we have connected a wire from the output, n_{out}, to the negative input, n_-, and so those constitute one node with a single voltage, which we call v_{out}. We can write down the equations:

$$\begin{aligned} v_+ &= V_c \\ v_{out} - v_{gnd} &= K(v_+ - v_{out}) \\ v_{gnd} &= 0 \end{aligned}$$

Solving this system, we find that

$$v_{out} = V_c \frac{K}{K+1} \quad .$$

So, for large K, the voltage difference between the output terminals is very close to the voltage difference between the input terminals.

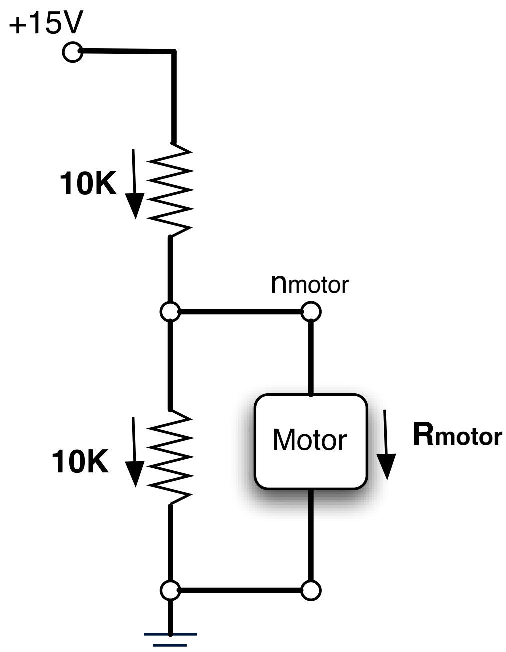
In this circuit we have set $v_{gnd} = 0$; in practice this doesn't matter. If we had left v_{gnd} in our equations, we would have found that:

$$v_{out} = V_c \frac{K}{K+1} + v_{gnd} \frac{1}{K+1} \quad .$$

Since K is very large, that second term can be ignored. In fact, if you look at an actual op-amp, you won't even find an output corresponding to n_{gnd}. Therefore, we will always treat it as being zero.

The crucial additional fact about an op-amp is that no current flows between the input and output terminals, so the op-amp acts as a *buffer* between the input and output signal. It provides a disconnect between the currents on either side of it. This is a big deal: it gives us a kind of modularity in our circuits that we haven't had before, by limiting the kinds of influence of one part of the circuit on the other. The ability to partially disconnect sub-parts of our circuits will make it easier to do complex designs.

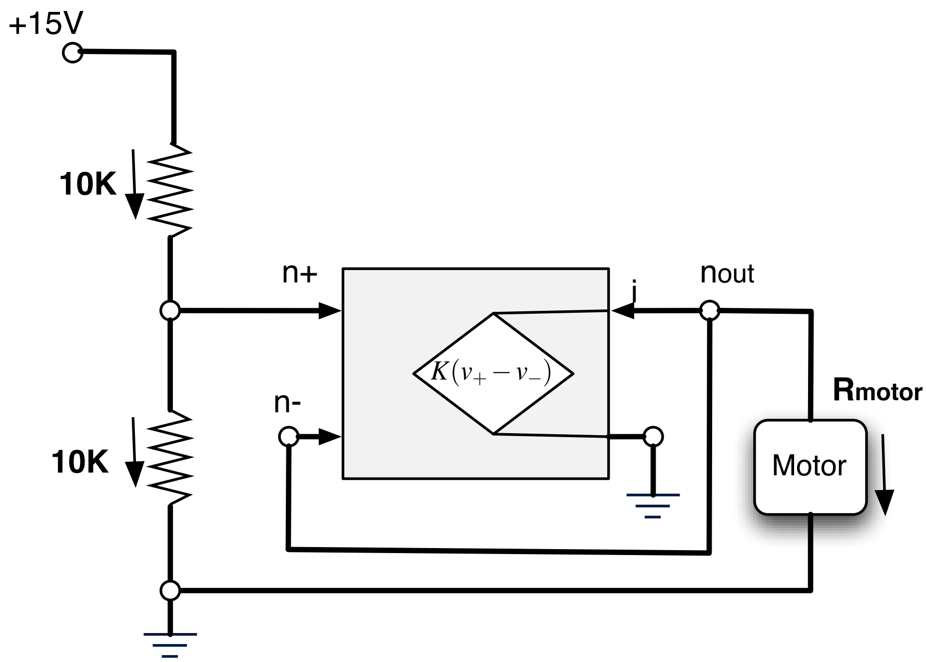
We can appreciate the value of a buffer in the context of using a variable voltage to control a motor. If we have a 15V supply, but only want to put 7.5V across the motor terminals, what should we do? A voltage divider seems like a good strategy: we can use one with two equal resistances, to make 7.5V, and then connect it to the motor as shown here



But what will the voltage v_{motor} end up being? It all depends on the resistance of the motor. If the motor is offering little resistance, say 100Ω, then the voltage v_{motor} will be very close to 0.[43] So, this is not an effective solution to the problem of supplying 7.5V to the motor.

So, now, back to the motor. In this figure,

[43] Go back and review the discussion of adding a load to a voltage divider, if this doesn't seem clear.



we have used a voltage follower to connect the voltage divider to the motor. Based on our previous analysis of the follower, we expect the voltage at n_{out} to be 7.5V, at least before we connect it up to the motor. Let's see how this all works out by writing the equations out for the whole circuit: Letting the current through the top resistor be i_1, the current through the second resistor be i_2, the current through the motor be i_m, and recalling that $v_{out} = v_-$ (since they are the same node), we have

$$
\begin{aligned}
i_1 - i_2 - i_+ &= 0 \\
-i_{out} - i_- - i_m &= 0 \\
i_+ &= 0 \\
i_- &= 0 \\
(15 - v_+) &= 10000 i_1 \\
(v_+ - 0) &= 10000 i_2 \\
(v_- - 0) &= K(v_+ - v_-) \\
(v_- - 0) &= i_m R_m
\end{aligned}
$$

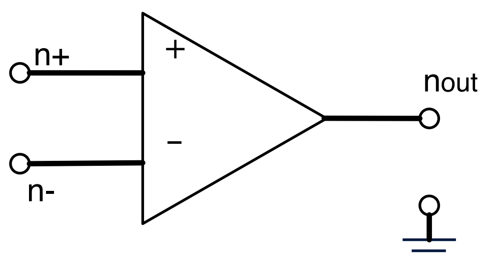
Solving these equations, we find that

$$
\begin{aligned}
v_+ &= 7.5 \\
v_- &= \frac{K}{K+1} 7.5 \\
i_m &= \frac{K}{K+1} \frac{7.5}{R_m}
\end{aligned}
$$

So, now, the voltage across the motor stays near $7.5V$, and because of the isolation provided by the op-amp, it will remain that way, no matter what the resistance of the motor. Further, if in fact the motor has resistance of 5Ω, then we can find that the (maximum) current through the motor, i_m, is approximately $1.5A$.

6.6.2 Simplified model

The voltage-controlled voltage-source model allows us to use our standard linear-equation form of reasoning about circuit values, but it can become cumbersome to deal with the K's all the time. In most of the usual applications of an op-amp, we will find that we can use a simpler model to reason about what is happening. That simpler model also comes with a simpler picture:

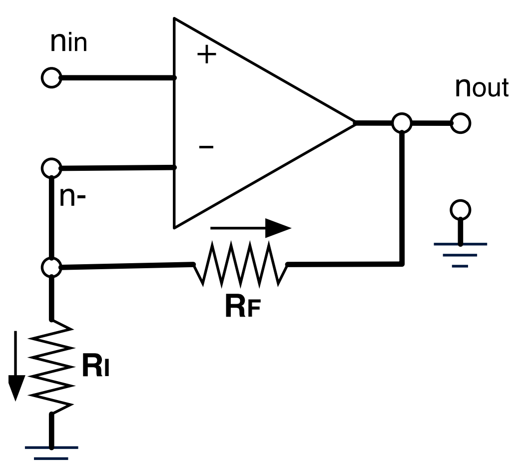


The simplified behavioral model is that the op-amp adjusts v_{out} in order to try to maintain the constraint that $v_+ \approx v_-$ and that no current flows in to n_+ or n_-. In what follows we will assume that the op-amp enforces $v_+ = v_-$.

In this section, we will explore several standard applications of op-amps, using this simplified model. If it becomes confusing, you can always go back and apply the VCVS model from the previous section.

Non-inverting amplifier

Not surprisingly, a primary use of an op-amp is as an amplifier. Here is an amplifier configuration, like this.



Let's see if we can figure out the relationship between v_{in} and v_{out}. The circuit constraints tell us that

$$v_- = i_I R_I$$

$$v_- - v_{out} = i_F R_F$$

$$-i_I - i_F = 0 \tag{6.3}$$

$$v_{in} = v_- \tag{6.4}$$

The KCL equation (6.3) has no term for the current into the op-amp, because we assume it is zero. Equation 6.4 is the op-amp constraint. So, we find that

$$v_{out} = v_{in}\frac{R_F + R_I}{R_I} \quad .$$

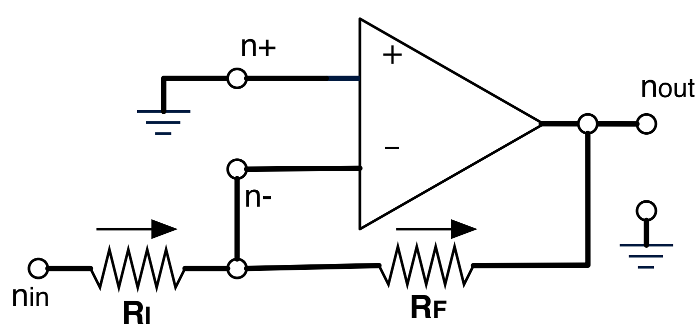
This is cool. We've arranged for the output voltage to be greater than the input voltage. We can get any gain (greater than one) by choosing the values of R_F and R_I.

We can think intuitively about how it works by examining some cases. First, if $R_F = 0$, then we'll have $v_{out} = v_{in}$, so there's not a particularly interesting change in the voltages. This is still a useful device, essentially the voltage follower, which we have already seen, with an additional resistor. This extra resistor does not change the voltage relationsips but does increase the current flowing from the op-amp. Note also that choosing R_I very much larger than R_F also gives us a follower.

Now let's think about a more interesting case, but simplify matters by setting $R_F = R_I$. We can look at the part of the circuit running from V_{out} through R_F and R_I to ground. This looks a lot like a voltage divider, with v_- coming out of the middle of it. Because v_- needs to be the same as v_{in}, and it is v_{out} being divided in half, then v_{out} clearly has to be $2v_{in}$.

Inverting amplifier

Here is a very similar configuration, called an *inverting amplifier*.



The difference is that the + terminal of the op-amp is connected to ground, and the we're thinking of the path through the resistors as the terminal of the resulting circuit. Let's figure out the relationship between v_{in} and v_{out} for this one. The circuit constraints tell us that

$$v_{in} - v_- = i_I R_I$$
$$v_- - v_{out} = i_F R_F$$
$$i_I - i_F = 0$$
$$v_+ = v_-$$
$$v_+ = 0$$

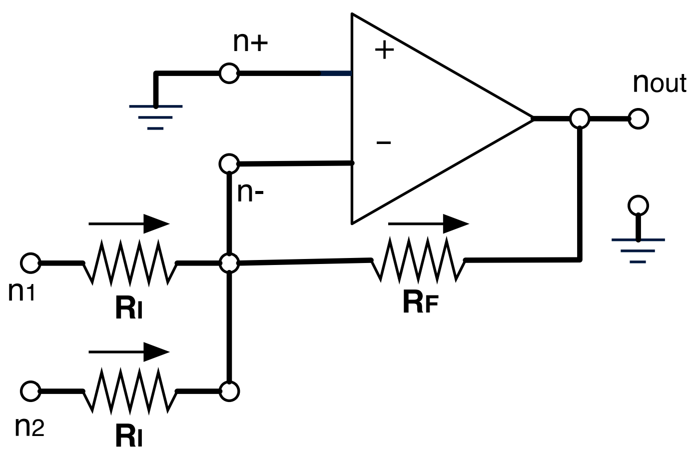
Solving, we discover that

$$v_{out} = -v_{in} \frac{R_F}{R_I} \ .$$

If $R_F = R_I$, then this circuit simply inverts the incoming voltage. So, for example, if v_{in} is $+10V$ with respect to ground, then v_{out} will be $-10V$. Again, we can see the path from n_{in} through the resistors, to n_{out}, as a voltage divider. Knowing that v_- has to be 0, we can see that v_{out} has to be equal to $-v_{in}$. If we want to scale the voltage, as well as invert it, we can do that by selecting appropriate values of R_F and R_I.

Voltage summer

A *voltage summer*[44] circuit, shown here, can be thought of as having three terminals, with the voltage at n_{out} constrained to be a scaled, inverted, sum of the voltages at n_1 and n_2.



You should be able to write down the equations for this circuit, which is very similar to the inverting amplifier, and derive the relationship:

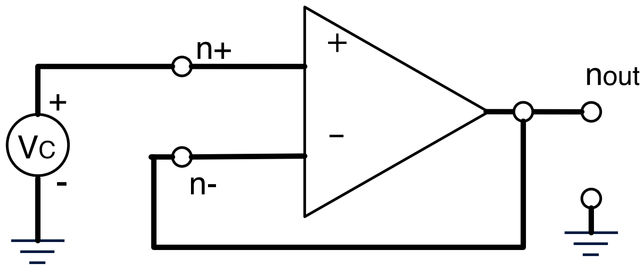
$$v_{out} = -\frac{R_F}{R_I}(v_1 + v_2) \ .$$

6.6.3 Op-amps and feedback

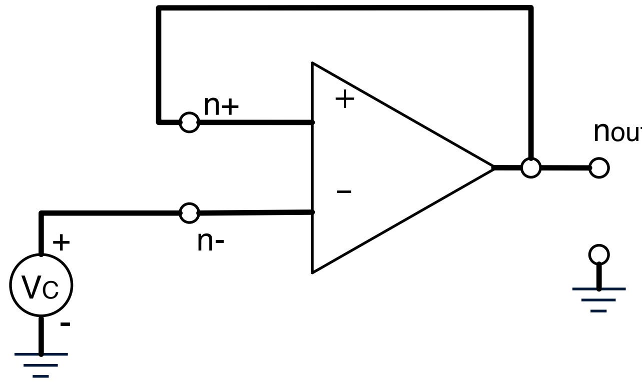
We have now seen two different models of op-amps. Each of those models can be useful in some circumstances, but now we'll explore some situations that require us to use an even more detailed and faithful model of how an op-amp really behaves.

44 As in thing that sums, not as in endless summer.

Here are two voltage followers, a "good" one and a "bad" one.



"Good" voltage follower



"Bad" voltage follower

If we use our simplest model to predict their behavior, in both cases, we'll predict that $v_{out} = V_c$. If we use the VCVS model, we'll predict that, for the good follower $v_{out} = V_c \frac{K}{K+1}$. For the bad follower, things are connected a little bit differently:

$$
\begin{aligned}
v_- &= V_c \\
v_{out} - v_{gnd} &= K(v_+ - v_-) \\
v_+ &= v_{out} \\
v_{gnd} &= 0
\end{aligned}
$$

Solving this system, we find that

$$v_{out} = V_c \frac{K}{K-1} \quad .$$

Those two predictions are basically the same for large K. But, in fact, the prediction about the behavior of the bad follower is **completely bogus**! In fact, if you connect an op-amp this way, it is likely to burn out, possibly generating smoke and noise in the process.

To see why this happens, we'll actually need to make a more detailed model.

Dynamics model

Our new model is going to take the actual dynamics of the circuit into account. We can model what is going on in an op-amp by using a difference equation to describe the value of the output at a time n as a linear function of its values at previous times and of the values of v_+ and v_-.

Note that we're using $v_o = v_{out}$ in the following.

You might imagine that the way the op-amp 'tries' to make V_o equal $K(V_+ - V_-)$ is by gradually adjusting V_o to decrease the error, E, making adjustments that are proportional, with a positive 'gain' g, to the error. That is, that

$$
\begin{aligned}
v_o[t] &= v_o[t-1] - g \cdot e[t-1] \\
&= v_o[t-1] - g \cdot (v_o[t-1] - K(v_+[t-1] - v_-[t-1])) \qquad (6.5)
\end{aligned}
$$

We would expect this system to adjust v_o to be closer to $K(v_+ - v_-)$ on every step. The gain g is a property of the design of the op-amp.

Now, what happens when we use this model to predict the temporal behavior of the voltage followers? Let's start with the good one.

Good follower

In this case, we have $v_-[t] = v_o[t]$ for all t. So, **equation 6.5** becomes

$$\begin{aligned} v_o[t] &= v_o[t-1] - g \cdot (v_o[t-1] - K(v_+[t-1] - v_o[t-1])) \\ &= (1 - g - gK)v_o[t-1] + gKv_+[t-1] \end{aligned}$$

Writing this as an operator equation, and then a system function, we get

$$\begin{aligned} V_o &= (1 - g - gK)\mathcal{R}V_o + gK\mathcal{R}V_+ \\ V_o(1 - \mathcal{R}(1 - g - gK)) &= gK\mathcal{R}V_+ \\ \frac{V_o}{V_+} &= \frac{gK\mathcal{R}}{1 - \mathcal{R}(1 - g - gK)} \end{aligned}$$

This system has a pole at $1 - g - gK$, so if we let $g = 1/(1 + K)$, then we have a pole at 0, which means the system will, ideally, converge in a single step, with

$$V_o = \frac{K}{K+1}\mathcal{R}V_+ \quad ,$$

which agrees with our previous model.

Bad follower

In this case, we have $v_+[t] = v_o[t]$ for all t. This time, **equation 6.5** becomes

$$\begin{aligned} v_o[t] &= v_o[t-1] - g \cdot (v_o[t-1] - K(v_o[t-1] - v_-[t-1])) \\ &= (1 - g + gK)v_o[t-1] - gKv_-[t-1] \end{aligned}$$

Writing this as an operator equation, and finding the system function, we get

$$\begin{aligned} V_o &= (1 - g + gK)\mathcal{R}V_o - gK\mathcal{R}V_- \\ V_o(1 - \mathcal{R}(1 - g + gK)) &= -gK\mathcal{R}V_- \\ \frac{V_o}{V_-} &= \frac{-gK\mathcal{R}}{1 - \mathcal{R}(1 - g + gK)} \end{aligned}$$

This system has a pole at $1 - g + gK$, so with $g = 1/(1+K)$ as before, we have a pole at $2K/(1+K)$, which means the system will diverge for large positive values of K. There is no positive value of g that will make this system stable (and, if we were to pick a negative value, it would make the good follower become a bad follower).

The prediction we get for $g = 1/(1+K)$ agrees with the empirical behavior of op-amps, at least in so far as op-amps wired up in feedback to the postive input terminal do not converge to a stable output voltage, but typically burn themselves out, or behave otherwise unpredictably.

6.6.4 Where does the current come from?

In all of the models we have seen so far, we've left out an important point. As you know from lab, op-amps, in fact, need two more connections, which is to V_{CC}, the main positive voltage supply, and to $V_{EE/GND}$, which is the negative power supply (which can be ground). Why? If no current flows "through" the op-amp, from n_+ or n_- to the other side, then how can it maintain the necessary voltage difference on the output side, and supply current to, e.g., drive the motor? The answer is that it uses current from the power supplies to regulate the output voltage and to supply the necessary output current.

One metaphorical way of thinking about this is that V_{CC} is a big main water-supply pipe and that $v_+ - v_-$ is the controlling the handle of a faucet. As we change the size of the difference $v_+ - v_-$, we open or close the faucet; but the water is being supplied from somewhere else entirely.

This metaphor is appropriate in another way: the total voltage on the output of the op-amp is limited by V_{CC}. So, if the $V_{CC} = 10$, then the output of the op-amp is limited to be between 0 and 10. If we connect the ground connection of the op-amp to a negative supply voltage, then the output of the op-amp can be negative as well as positive; this has many applications.