6.01: Introduction to EECS I

Designing Control Systems

March 8, 2011

Midterm Examination #1

Time:	Tonight, March 8, 7:30 PM to 9:30 PM
Location:	Walker Memorial (if last name starts with A-M) 10-250 (if last name starts with N-Z)
Coverage:	Everything up to and including Design Lab 5.

You may refer to any printed materials that you bring to exam.

You may use a calculator.

You may not use a computer, phone, or music player.

No software lab this week.

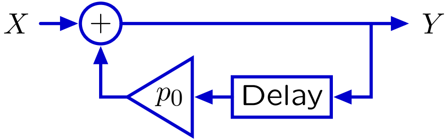
Signals and Systems

Multiple representations of systems, each with particular strengths.

Difference equations are mathematically compact.

$$y[n] = x[n] + p_0\, y[n-1]$$

Block diagrams illustrate signal flow paths from input to output.



Operators use polynomials to represent signal flow compactly.

$$Y = X + p_0 \mathcal{R} Y$$

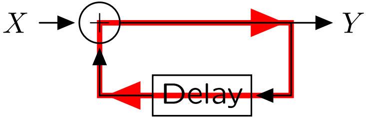
System Functionals represent systems as operators.

$$Y = H\,X \;\; ; \qquad H = \frac{Y}{X} = \frac{1}{1 - p_0 \mathcal{R}}$$

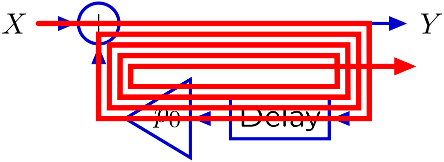
Feedback, Cyclic Signal Paths, and Poles

The structure of feedback produces characteristic behaviors.

Feedback produces cyclic signal flow paths.

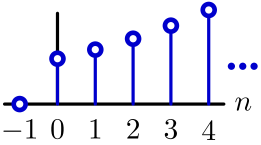


Cyclic signal flow paths → persistent responses to transient inputs.



We can characterize persistent responses (called modes) with poles.

$$y[n] = p_o^n\,;\;\; n \geq 0$$

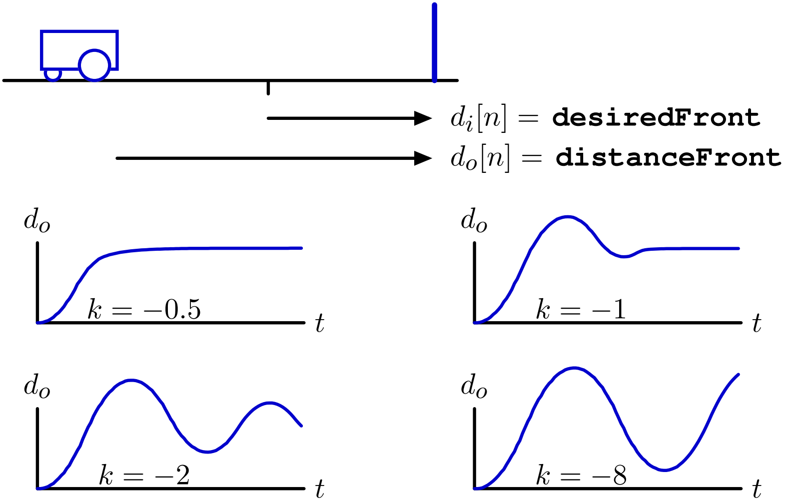


Designing a Control System

Today's goal: optimizing the design of a control system.

Example: wallFinder System

Using feedback to control position (lab 4) can lead to bad behaviors.

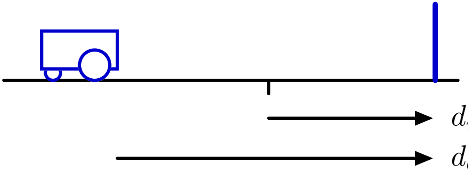


What causes these different types of responses?

Is there a systematic way to optimize the gain k?

Analysis of wallFinder System: Review

Response of system is concisely represented with difference equation.



$d_i[n] =$ **`desiredFront`**

$d_o[n] =$ **`distanceFront`**

proportional controller: $v[n] = ke[n] = k\big(d_i[n] - d_s[n]\big)$

locomotion: $d_o[n] = d_o[n-1] - Tv[n-1]$

sensor with no delay: $d_s[n] = d_o[n]$

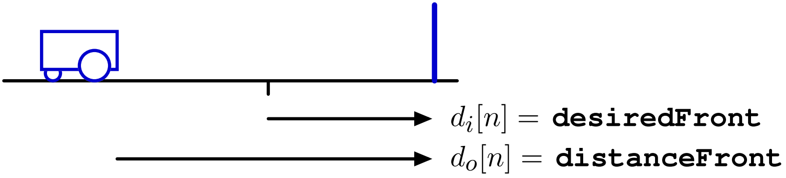
The difference equations provide a concise description of behavior.

$$d_o[n] = d_o[n-1] - Tv[n-1] = d_o[n-1] - Tk\big(d_i[n-1] - d_o[n-1]\big)$$

However it provides little insight into how to choose the gain k.

Analysis of wallFinder System: Block Diagram

A block diagram for this system reveals two feedback paths.



proportional controller: $v[n] = ke[n] = k\big(d_i[n] - d_s[n]\big)$

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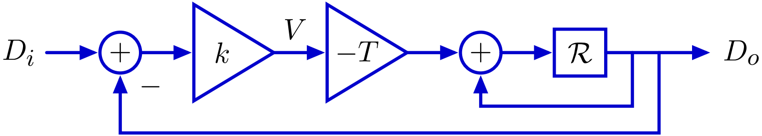
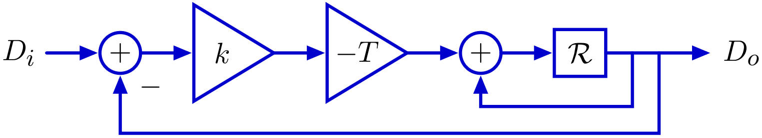


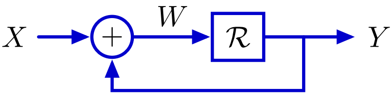
Analysis of wallFinder System: System Functions

Simplify block diagram with $\mathcal{R}$ operator and system functions.

Start with accumulator.



What is the input/output relation for an accumulator?



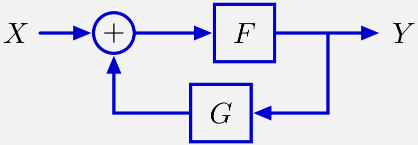
$$Y = \mathcal{R}W = \mathcal{R}(X + Y)$$

$$\frac{Y}{X} = \frac{\mathcal{R}}{1 - \mathcal{R}}$$

This is an example of a recurring pattern: **Black's equation**.

Check Yourself

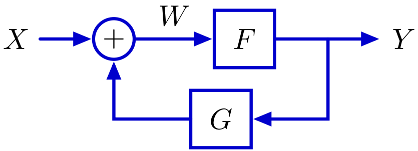
Determine the system function $H = \frac{Y}{X}$.



1. $\frac{F}{1 - FG}$
2. $\frac{F}{1 + FG}$
3. $F + \frac{1}{1 - G}$
4. $F \times \frac{1}{1 - G}$
5. none of the above

Black's Equation

Determine the system function $H = \frac{Y}{X}$.



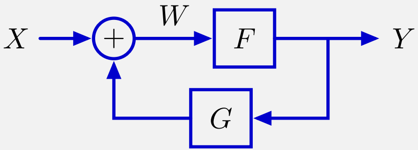
$$Y = FW = F(X + GY) = FX + FGY$$

$$\frac{Y}{X} \equiv H = \frac{F}{1 - FG}$$

closed-loop gain $H = \frac{\textbf{forward gain } F}{1 - \textbf{loop gain } FG}$

Check Yourself

Determine the system function $H = \frac{Y}{X}$. 1



1. $\frac{F}{1 - FG}$
2. $\frac{F}{1 + FG}$
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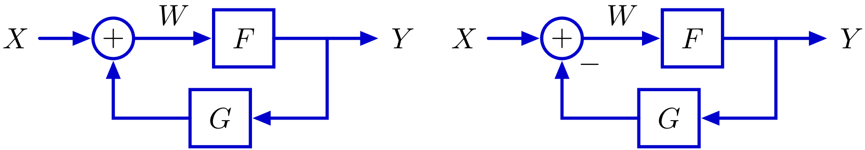
4. $F \times \frac{1}{1 - G}$
5. none of the above

Black's Equation

Black's equation has two common forms.



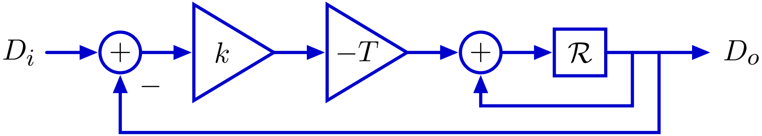
$$H = \frac{F}{1 - FG} \qquad H = \frac{F}{1 + FG}$$

Difference is equivalent to changing sign of G.

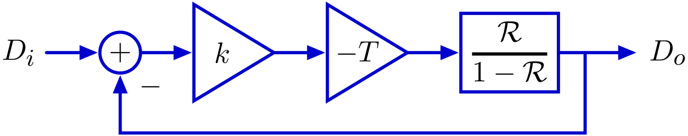
Right form is useful in most **control** applications where the goal is to make Y converge to X.

Analyzing wallFinder: System Functions

Simplify block diagram with $\mathcal{R}$ operator and system functions.



Replace accumulator with equivalent block diagram.

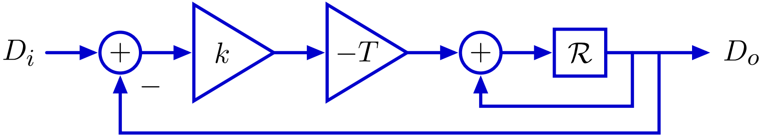


Now apply Black's equation a second time:

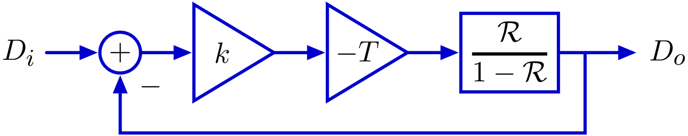
$$\frac{D_o}{D_i} = \frac{\frac{-kT\mathcal{R}}{1-\mathcal{R}}}{1+\frac{-kT\mathcal{R}}{1-\mathcal{R}}} = \frac{-kT\mathcal{R}}{1-\mathcal{R}-kT\mathcal{R}} = \frac{-kT\mathcal{R}}{1-(1+kT)\mathcal{R}}$$

Analyzing wallFinder: System Functions

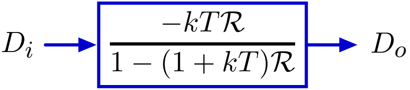
We can represent the entire system with a single system function.



Replace accumulator with equivalent block diagram.



Equivalent system with a single block:



Modular! But we still need a way to choose k.

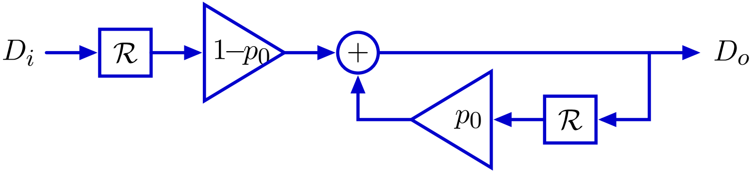
Analyzing wallFinder: Poles

The system function contains a single **pole** at $z = 1 + kT$.

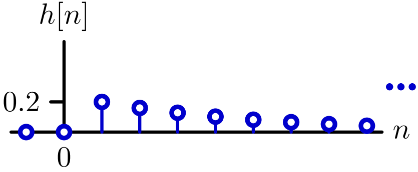
$$\frac{D_o}{D_i} = \frac{-kT\mathcal{R}}{1 - (1 + kT)\mathcal{R}}$$

The numerator is just a gain and a delay.

The whole system is equivalent to the following:

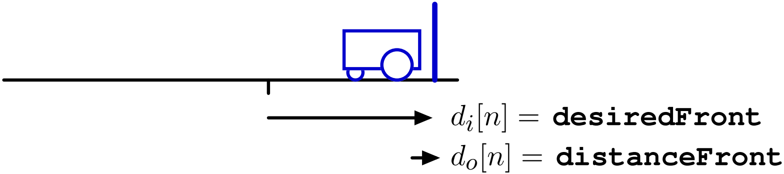


where $p_o = 1 + kT$. Here is the unit-sample response for $kT = -0.2$:



Analyzing wallFinder

We are often interested in the **step response** of a control system.

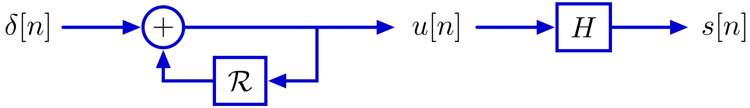


Start the output $d_o[n]$ at zero while the input is held constant at one.

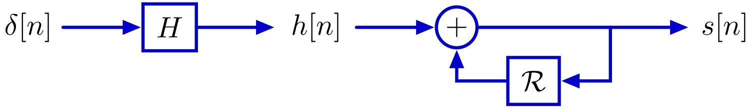
Step Response

Calculating the unit-step response.

Unit-step response $s[n]$ is response of H to the unit-step signal $u[n]$, which is constructed by accumulation of the unit-sample signal $\delta[n]$.



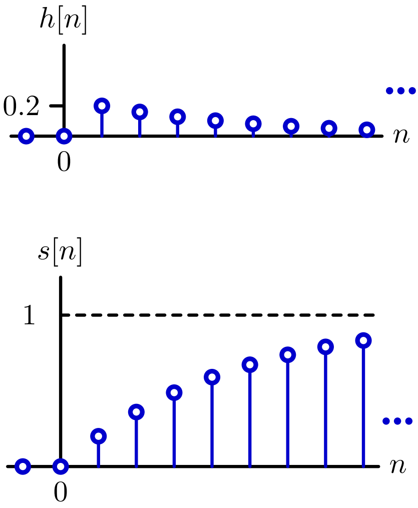
Commute and relabel signals.



The unit-step response $s[n]$ is equal to the accumulated responses to the unit-sample response $h[n]$.

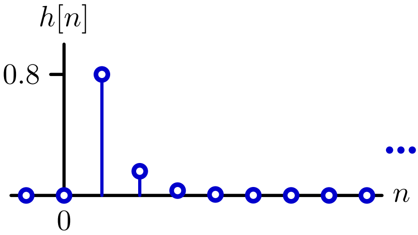
Analyzing wallFinder

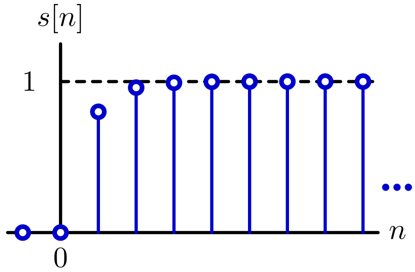
The step response of the wallFinder system is slow because the unit-sample response is slow.



Analyzing wallFinder

The step response is faster if $kT = -0.8$ (i.e., $p_0 = 0.2$).

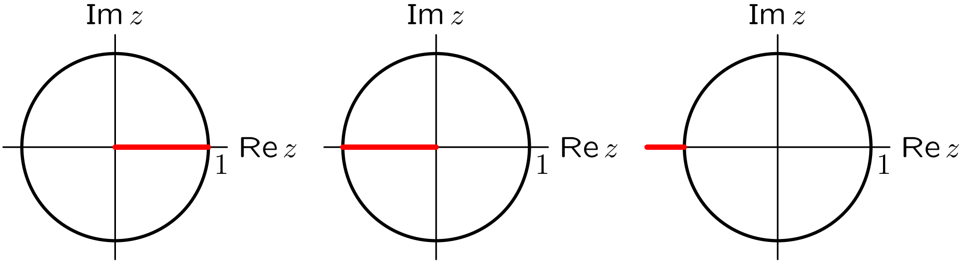




Analyzing wallFinder: Poles

The poles of the system function provide insight for choosing k.

$$\frac{D_o}{D_i} = \frac{-kT\mathcal{R}}{1-(1+kT)\mathcal{R}} = \frac{(1-p_o)\mathcal{R}}{1-p_o\mathcal{R}} \;; \qquad p_0 = 1 + kT$$



$-1 < kT < 0$	$-2 < kT < -1$	$kT < -2$
$0 < p_0 < 1$	$-1 < p_0 < 0$	$p_0 < -1$
monotonic	alternating	alternating
converging	converging	diverging

Check Yourself

Find kT for fastest convergence of unit-sample response.

$$\frac{D_o}{D_i} = \frac{-kT\mathcal{R}}{1-(1+kT)\mathcal{R}}$$

1. $kT = -2$
2. $kT = -1$
3. $kT = 0$
4. $kT = 1$
5. $kT = 2$
0. none of the above

Check Yourself

Find kT for fastest convergence of unit-sample response.

$$\frac{D_o}{D_i} = \frac{-kT\mathcal{R}}{1-(1+kT)\mathcal{R}}$$

If $kT = -1$ then the pole is at $z = 0$.

$$\frac{D_o}{D_i} = \frac{-kT\mathcal{R}}{1-(1+kT)\mathcal{R}} = \mathcal{R}$$

unit-sample response has a single non-zero output sample, at $n = 1$.

Check Yourself

Find kT for fastest convergence of unit-sample response. 2

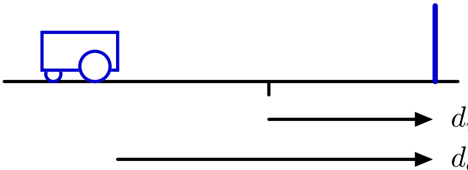
$$\frac{D_o}{D_i} = \frac{-kT\mathcal{R}}{1-(1+kT)\mathcal{R}}$$

1. $kT = -2$
2. $kT = -1$
3. $kT = 0$
4. $kT = 1$
5. $kT = 2$

0. none of the above

Analyzing wallFinder

The optimum gain k moves robot to desired position in **one** step.



$d_i[n]$ = **`desiredFront`**=1 m

$d_o[n]$ = **`distanceFront`**=2 m

$$kT = -1$$

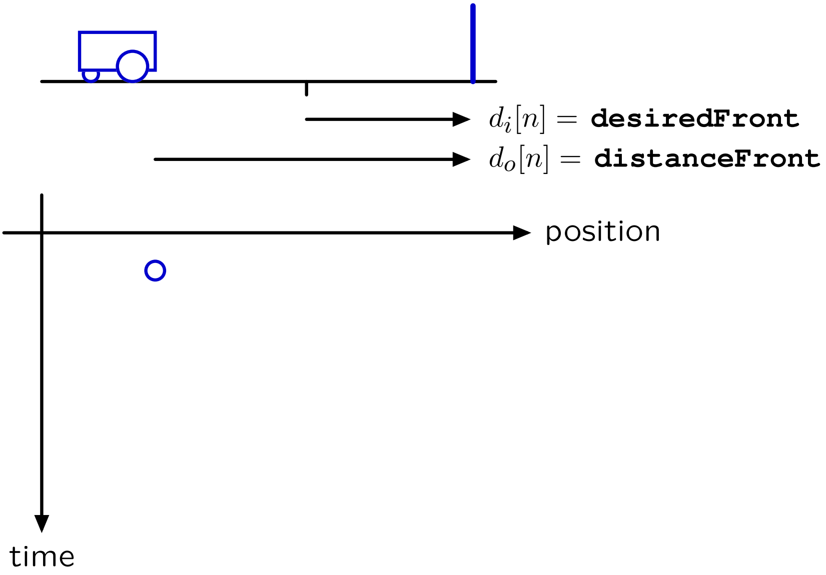
$$k = -\frac{1}{T} = -\frac{1}{1/10} = -10$$

$$v[n] = k\big(d_i[n] - d_o[n]\big) = -10\big(1 - 2\big) = 10 \text{ m/s}$$

exactly the right speed to get there in one step!

Analyzing wallFinder: Space-Time Diagram

The optimum gain k moves robot to desired position in **one** step.



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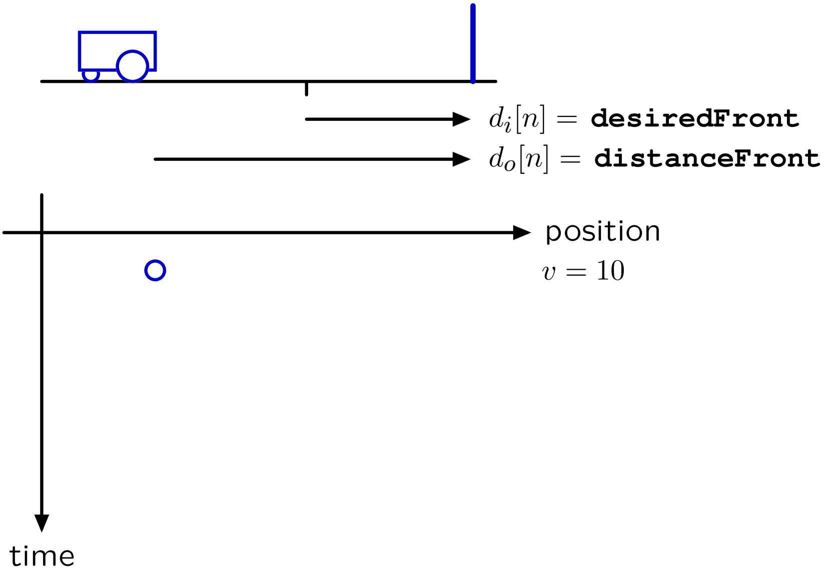

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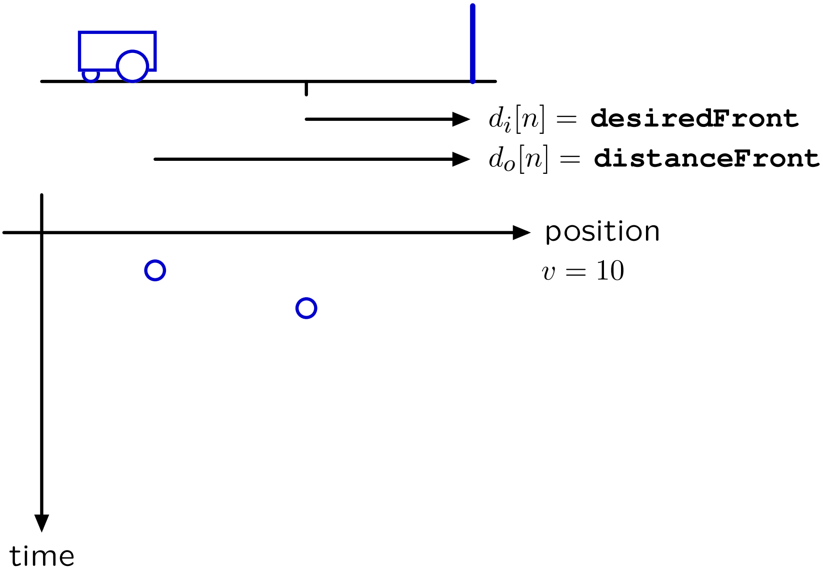

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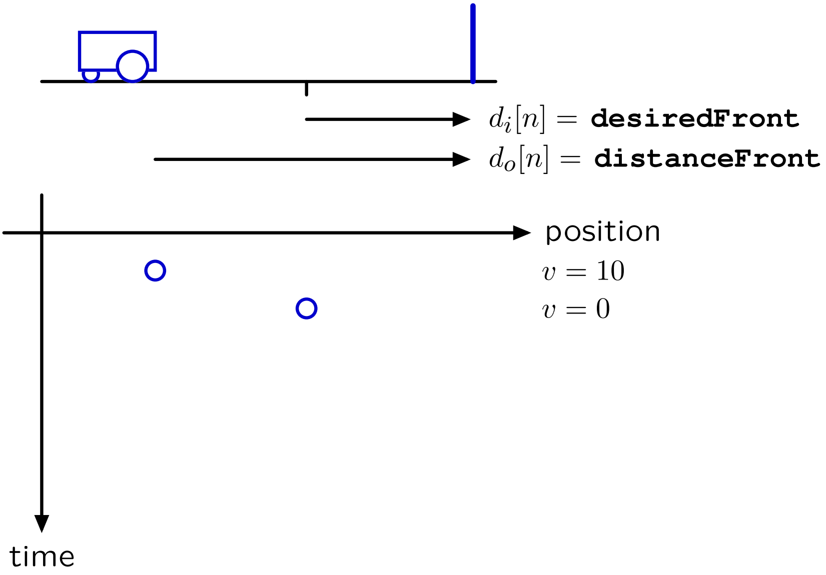

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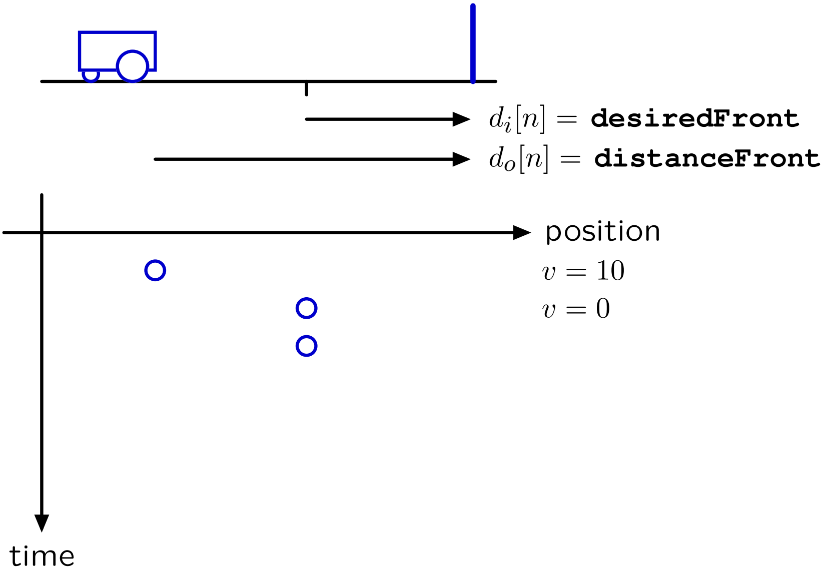
The optimum gain k moves robot to desired position in **one** step.



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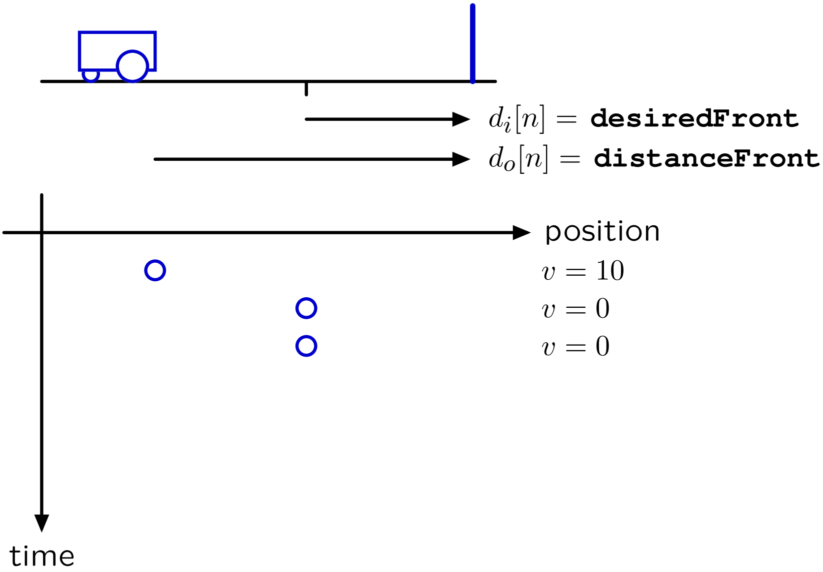

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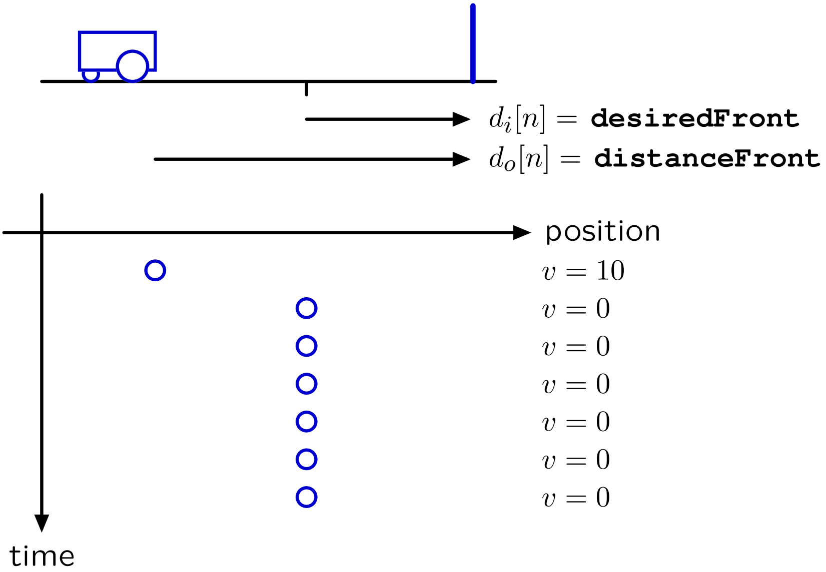
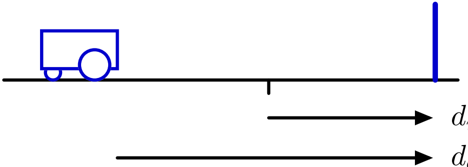


Analysis of wallFinder System: Adding Sensory Delay

Adding delay tends to destabilize control systems.



$d_i[n] = \texttt{desiredFront}$

$d_o[n] = \texttt{distanceFront}$

proportional controller: $v[n] = ke[n] = k\big(d_i[n] - d_s[n]\big)$

locomotion: $d_o[n] = d_o[n-1] - Tv[n-1]$

sensor **with delay**: $d_s[n] = d_o[\mathbf{n-1}]$

Analysis of wallFinder System: Adding Sensory Delay

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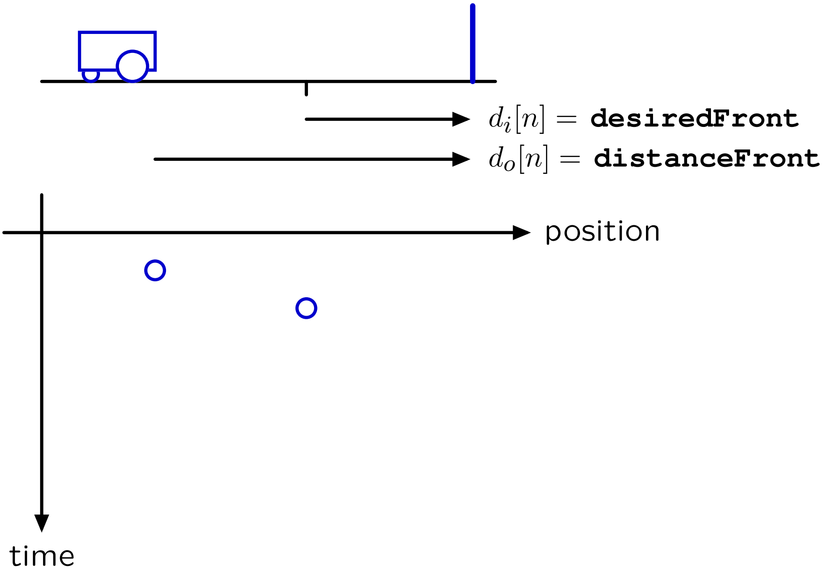

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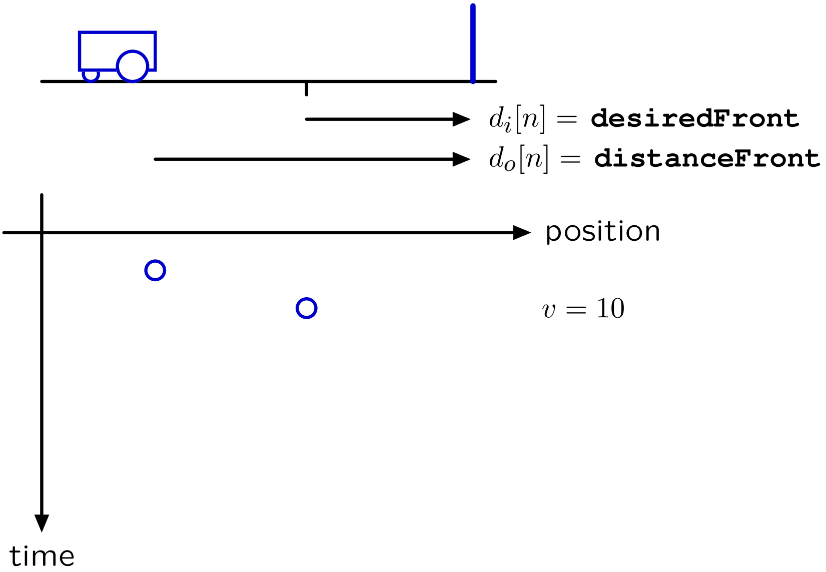

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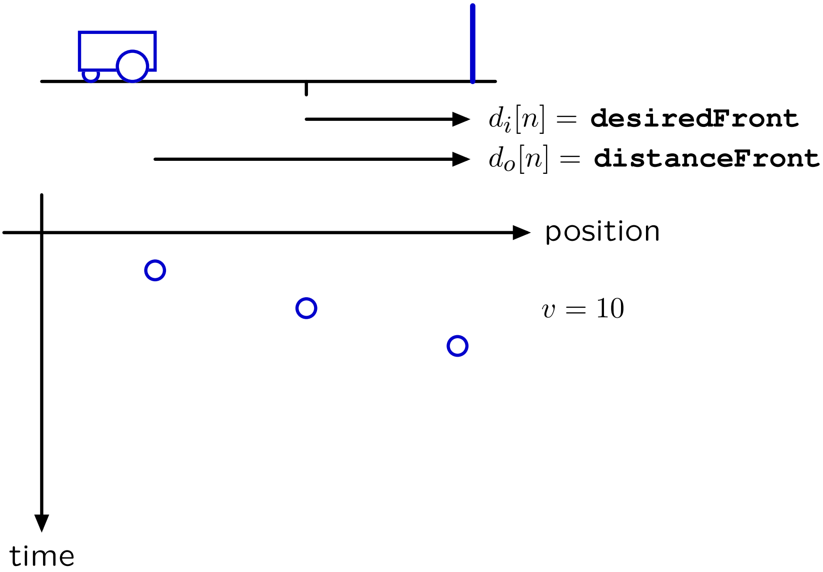

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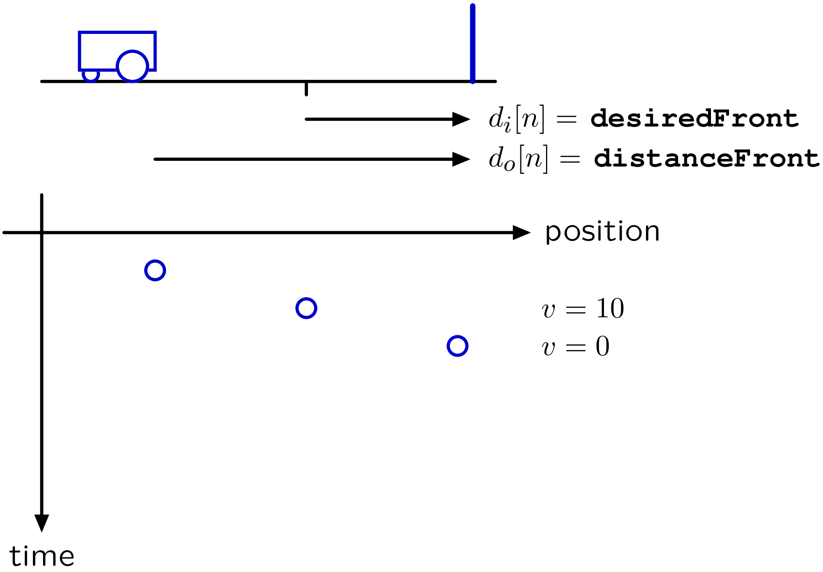

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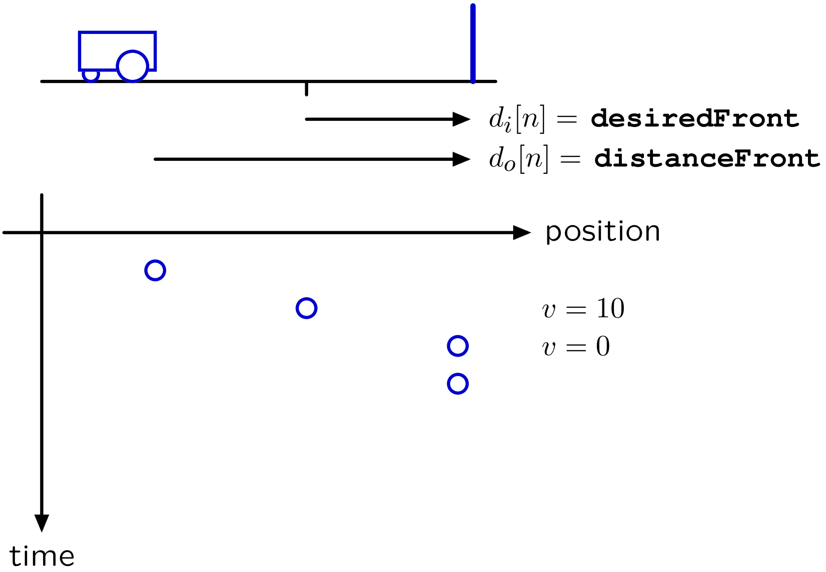

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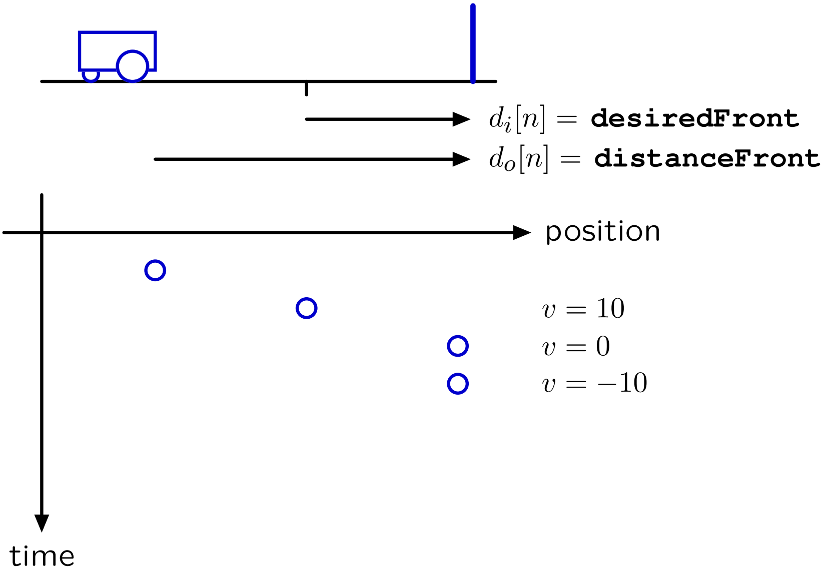
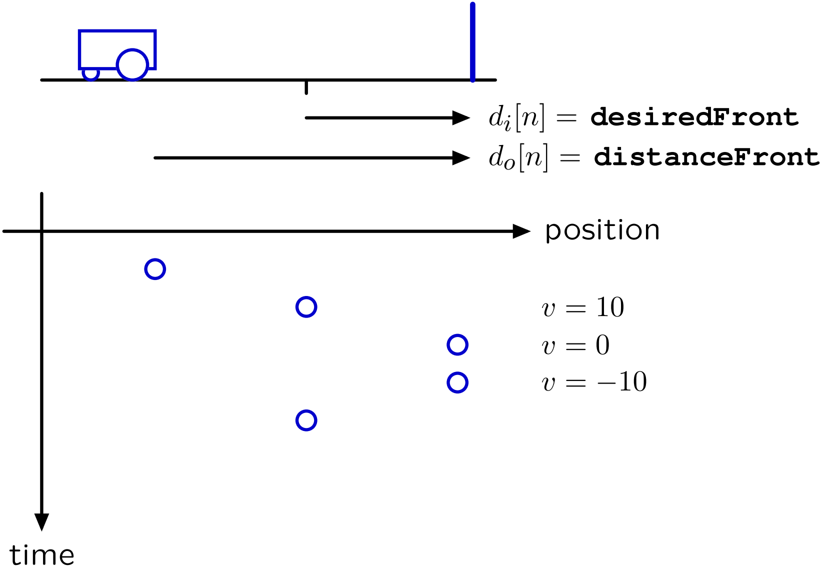


Analysis of wallFinder System: Adding Sensory Delay

Adding delay tends to destabilize control systems.



Analysis of wallFinder System: Adding Sensory Delay

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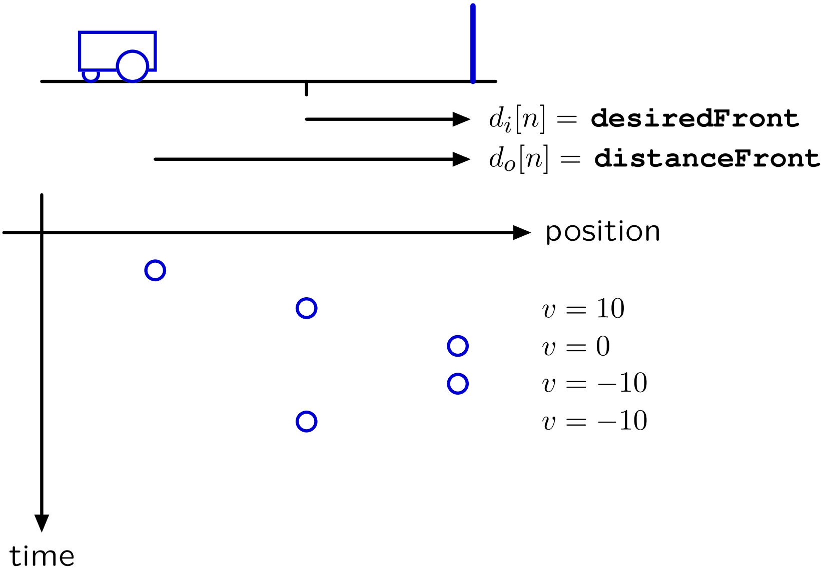

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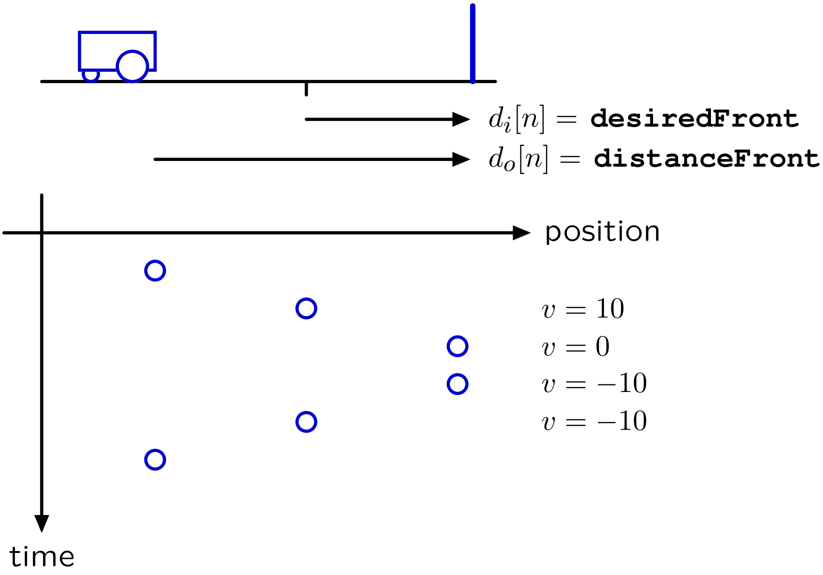

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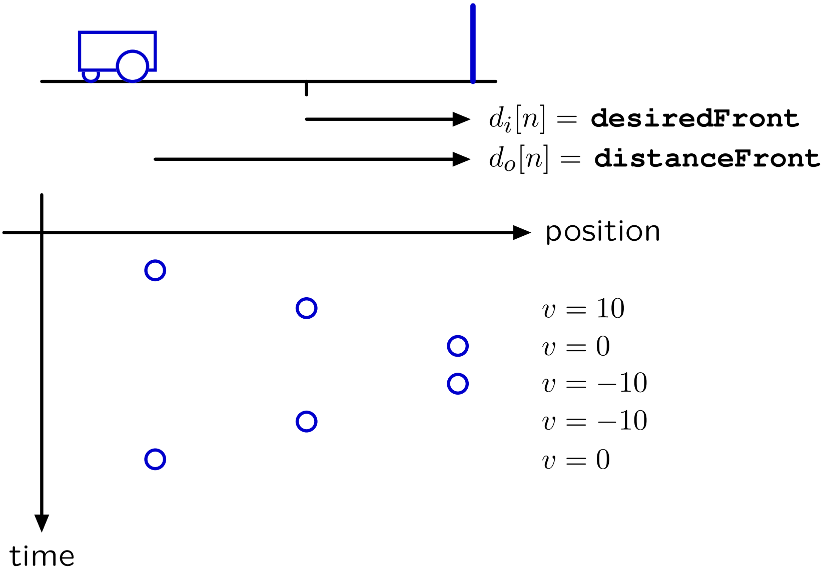

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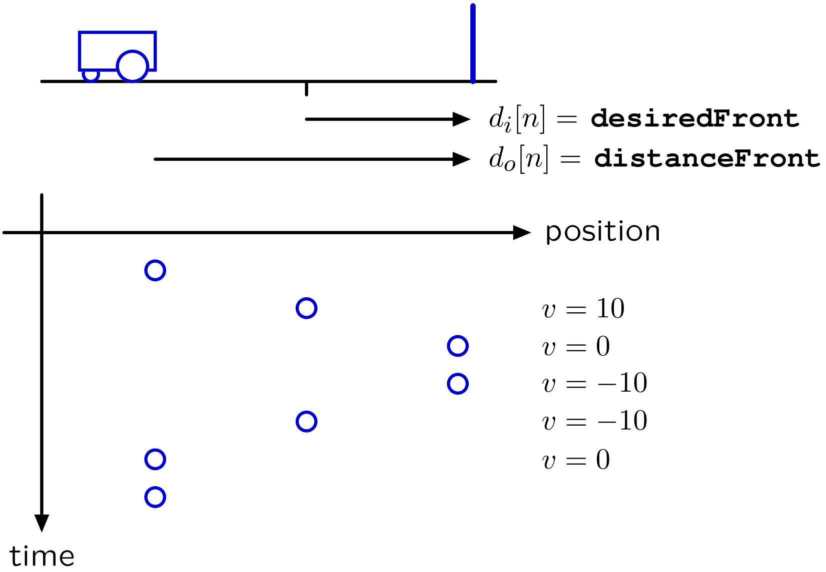
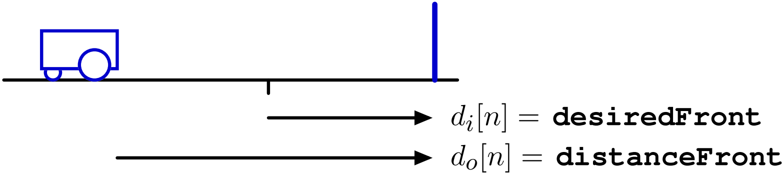


Analysis of wallFinder System: Block Diagram

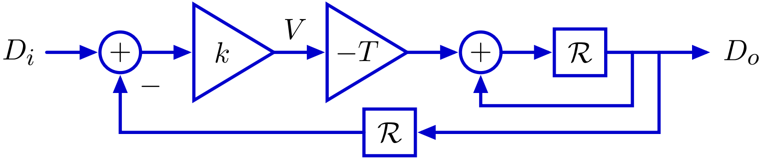
Incorporating sensor delay in block diagram.



proportional controller: $v[n] = ke[n] = k\big(d_i[n] - d_s[n]\big)$

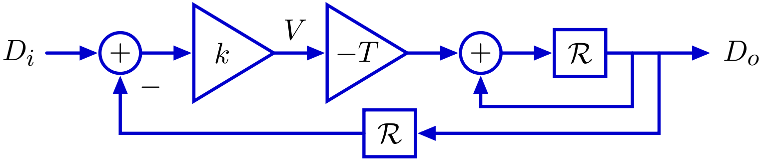
locomotion: $d_o[n] = d_o[n-1] - Tv[n-1]$

sensor with delay: $d_s[n] = d_o[n-1]$

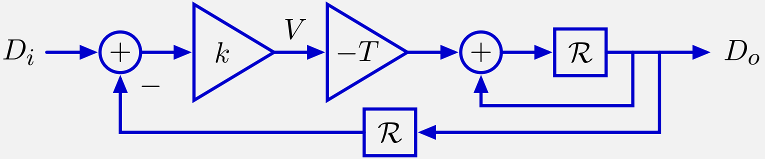


Analyzing wallFinder: System Functions

We can represent the entire system with a single system function.



Check Yourself

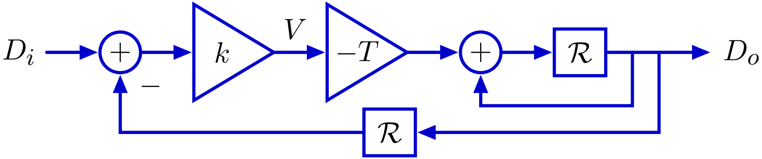


Find the system function $H = \dfrac{D_o}{D_i}$.

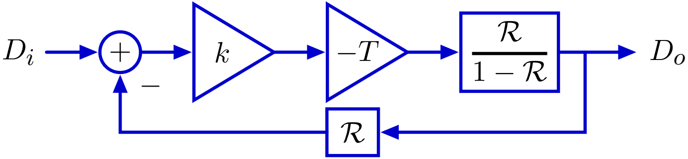
1. $\dfrac{kT\mathcal{R}}{1-\mathcal{R}}$
2. $\dfrac{-kT\mathcal{R}}{1+\mathcal{R}-kT\mathcal{R}^2}$
3. $\dfrac{kT\mathcal{R}}{1-\mathcal{R}} - kT\mathcal{R}$
4. $\dfrac{-kT\mathcal{R}}{1-\mathcal{R}-kT\mathcal{R}^2}$
5. none of the above

Check Yourself

Find the system function $H = \dfrac{D_o}{D_i}$.

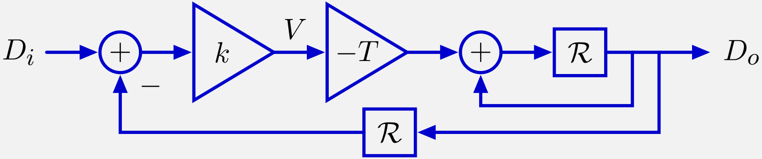


Replace accumulator with equivalent block diagram.



$$\frac{D_o}{D_i} = \frac{\dfrac{-kT\mathcal{R}}{1-\mathcal{R}}}{1 + \dfrac{-kT\mathcal{R}^2}{1-\mathcal{R}}} = \frac{-kT\mathcal{R}}{1-\mathcal{R}-kT\mathcal{R}^2}$$

Check Yourself



Find the system function $H = \frac{D_o}{D_i}$. 4

1. $\frac{kT\mathcal{R}}{1-\mathcal{R}}$
2. $\frac{-kT\mathcal{R}}{1+\mathcal{R}-kT\mathcal{R}^2}$
3. $\frac{kT\mathcal{R}}{1-\mathcal{R}} - kT\mathcal{R}$
4. $\frac{-kT\mathcal{R}}{1-\mathcal{R}-kT\mathcal{R}^2}$
5. none of the above

Analyzing wallFinder: Poles

Substitute $\frac{1}{z}$ for $\mathcal{R}$ in the system functional to find the poles.

$$\frac{D_o}{D_i} = \frac{-kT\mathcal{R}}{1 - \mathcal{R} - kT\mathcal{R}^2} = \frac{-kT\frac{1}{z}}{1 - \frac{1}{z} - kT\frac{1}{z^2}} = \frac{-kTz}{z^2 - z - kT}$$

The poles are then the roots of the denominator.

$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + kT}$$

Poles

Poles can be identified by expanding the system functional in partial fractions.

$$\frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \cdots}{1 + a_1\mathcal{R} + a_2\mathcal{R}^2 + a_3\mathcal{R}^3 + \cdots}$$

Factor denominator:

$$\frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \cdots}{(1 - p_0\mathcal{R})(1 - p_1\mathcal{R})(1 - p_2\mathcal{R})(1 - p_3\mathcal{R})\cdots}$$

Partial fractions:

$$\frac{Y}{X} = \frac{e_0}{1 - p_0\mathcal{R}} + \frac{e_1}{1 - p_1\mathcal{R}} + \frac{e_2}{1 - p_2\mathcal{R}} + \cdots + f_0 + f_1\mathcal{R} + f_2\mathcal{R}^2 + \cdots$$

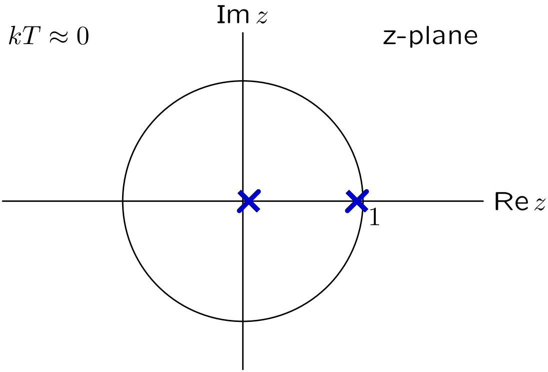
The poles are p_i for $0 \leq i < n$ where n is the order of the denominator.

One geometric mode p_i^n arises from each factor of the denominator.

Feedback and Control: Poles

If kT is small, the poles are at $z \approx -kT$ and $z \approx 1 + kT$.

$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + kT} = \frac{1}{2}\left(1 \pm \sqrt{1 + 4kT}\right) \approx \frac{1}{2}(1 \pm (1 + 2kT)) = 1 + kT,\ -kT$$



Pole near 0 generates fast response.

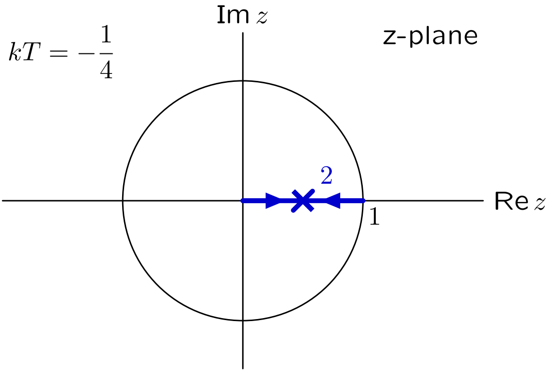
Pole near 1 generates slow response.

Slow mode (pole near 1) dominates the response.

Feedback and Control: Poles

As kT becomes more negative, the poles move toward each other and collide at $z = \frac{1}{2}$ when $kT = -\frac{1}{4}$.

$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + kT} = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - \frac{1}{4}} = \frac{1}{2}, \frac{1}{2}$$

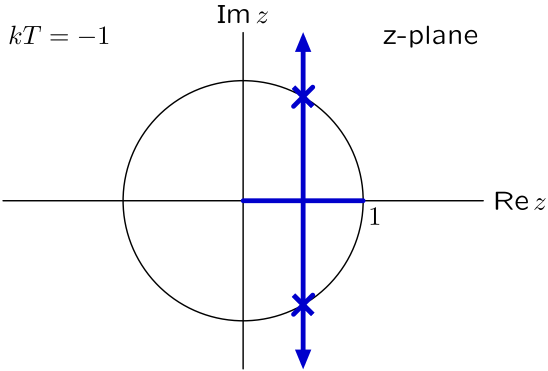


Persistent responses decay. The system is stable.

Feedback and Control: Poles

If $kT < -1/4$, the poles are complex.

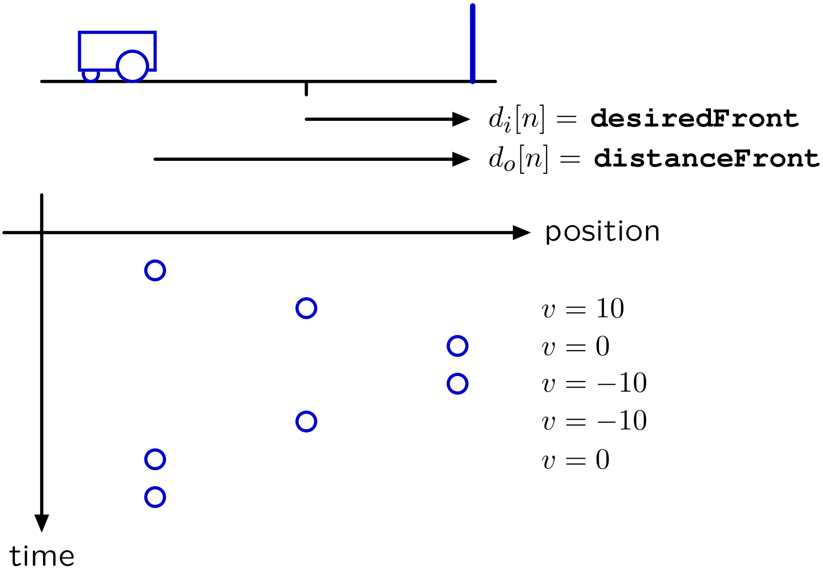
$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + kT} = \frac{1}{2} \pm j\sqrt{-kT - \left(\frac{1}{2}\right)^2}$$



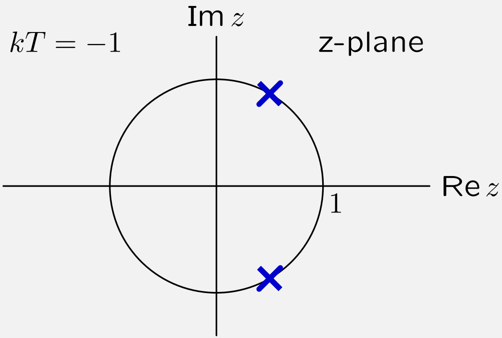
Complex poles → oscillations.

Same oscillation we saw earlier!

Adding delay tends to destabilize control systems.



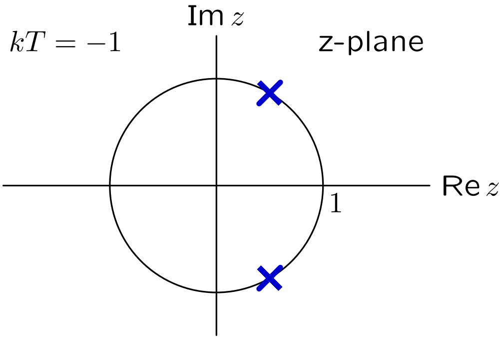
Check Yourself



What is the period of the oscillation?

1. 1
2. 2
3. 3
4. 4
5. 6
0. none of above

Check Yourself

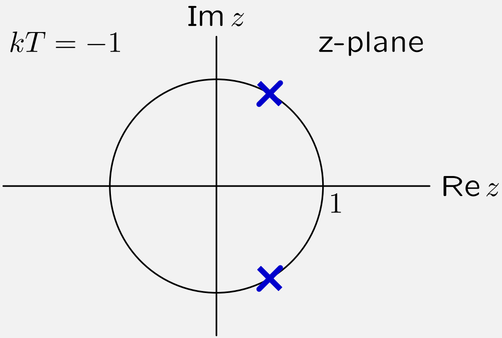


$$p_0=\frac{1}{2}\pm j\frac{\sqrt{3}}{2}=e^{\pm j\pi/3}$$

$$p_0^n=e^{\pm j\pi n/3}$$

$$\underbrace{e^{\pm j0\pi/3}}_{1},\ e^{\pm j\pi/3},\ e^{\pm j2\pi/3},\ e^{\pm j3\pi/3},\ e^{\pm j4\pi/3},\ e^{\pm j5\pi/3},\ \underbrace{e^{\pm j6\pi/3}}_{e^{\pm j2\pi}=1}$$

Check Yourself

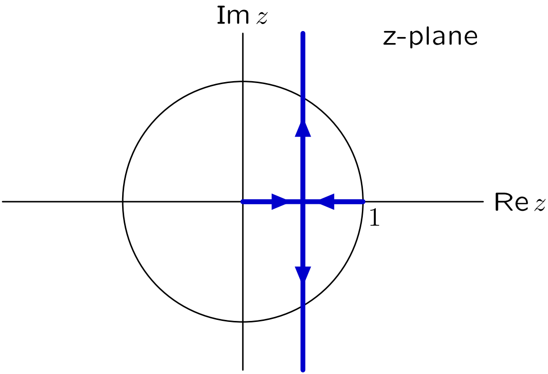


What is the period of the oscillation? 5

1. 1
2. 2
3. 3
4. 4
5. 6
0. none of above

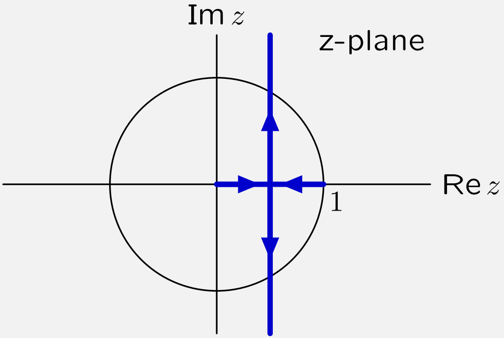
Feedback and Control: Poles

The closed-loop poles depend on the gain.



If $kT: 0 \to -\infty$: then $z_1, z_2 : 0, 1 \to \frac{1}{2}, \frac{1}{2} \to \frac{1}{2} \pm j\infty$

Check Yourself



closed-loop poles

$$\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + kT}$$

Find kT for fastest response.

1. 0 2. $-\frac{1}{4}$ 3. $-\frac{1}{2}$
4. -1 5. $-\infty$ 0. none of above

Check Yourself

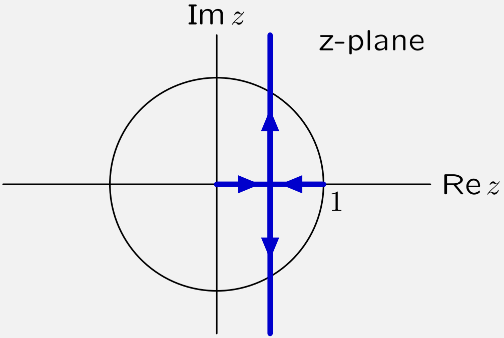
$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + kT}$$

The dominant pole always has a magnitude that is $\geq \frac{1}{2}$.

It is smallest when there is a double pole at $z = \frac{1}{2}$.

Therefore, $kT = -\frac{1}{4}$.

Check Yourself



closed-loop poles

$$\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + kT}$$

Find kT for fastest response. 2

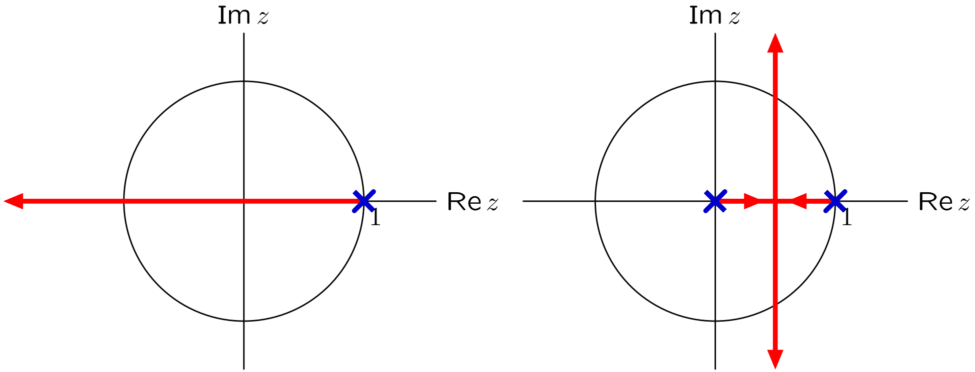
1. 0
2. $-\frac{1}{4}$
3. $-\frac{1}{2}$
4. -1
5. $-\infty$
0. none of above

Destabilizing Effect of Delay

Adding delay in the feedback loop makes it more difficult to stabilize.

Ideal sensor: $d_s[n] = d_o[n]$

More realistic sensor (with delay): $d_s[n] = d_o[n-1]$



Fastest response without delay: single pole at $z = 0$.

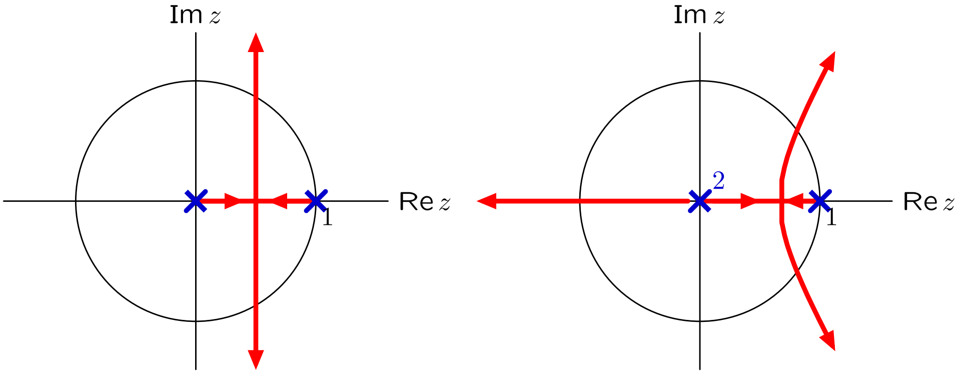
Fastest response with delay: double pole at $z = \frac{1}{2}$. **much slower!**

Destabilizing Effect of Delay

Adding more delay in the feedback loop is even worse.

More realistic sensor (with delay): $d_s[n] = d_o[n-1]$

Even more delay: $d_s[n] = d_o[n-2]$

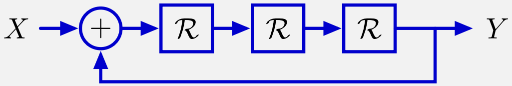


Fastest response with delay: double pole at $z = \frac{1}{2}$.

Fastest response with more delay: double pole at $z = 0.682$.

$\rightarrow$ **even slower**

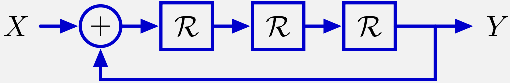
Check Yourself



How many of the following statements are true?

1. This system has 3 poles.
2. unit-sample response is the sum of 3 geometric sequences.
3. Unit-sample response is $y[n]: 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots$
4. Unit-sample response is $y[n]: 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots$
5. One of the poles is at $z = 1$.

Check Yourself



How many of the following statements are true? 4

1. This system has 3 poles.
2. unit-sample response is the sum of 3 geometric sequences.
3. Unit-sample response is $y[n] : 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots$
4. Unit-sample response is $y[n] : 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots$
5. One of the poles is at $z = 1$.

Designing Control Systems: Summary

System Functions provide a convenient summary of information that is important for designing control systems.

The long-term response of a system is determined by its dominant pole — i.e., the pole with the largest magnitude.

A system is unstable if the magnitude of its dominant pole is >1.

A system is stable if the magnitude of its dominant pole is <1.

Delays tend to decrease the stability of a feedback system.

6.01SC Introduction to Electrical Engineering and Computer Science

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