

## Problem Set 9 Solutions

### Problem 9.1

(a)

The waveguide will propagate only one mode in the frequency range between the two lowest cutoff frequencies. The cutoff frequency for the  $TE_{10}$  (which is the mode with the lowest cutoff frequency) is:

$$f_{10} = \sqrt{\left(\frac{c}{24}\right)^2} = 12.5 \times 10^6 \text{ [Hz]}$$

Using the result from part b below, we can say that between 12.5 MHz and 21.4 MHz only one mode will propagate. That mode is the  $TE_{10}$  mode.

(b)

Inside the waveguide, we expect the mode to follow the dispersion relation.

$$\left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2 + k_z^2 = \left(\frac{n\pi}{7}\right)^2 + \left(\frac{m\pi}{12}\right)^2 + k_z^2$$

In this case,  $c$  is the speed of light in air. The cutoff frequency is the frequency at which  $k_z$  goes from imaginary to real ( $k_z = 0$ ).

$$\left(\frac{2\pi f_{m,n}}{c}\right)^2 = \left(\frac{n\pi}{7}\right)^2 + \left(\frac{m\pi}{12}\right)^2$$

$$f_{m,n} = \sqrt{\left(m\frac{c}{24}\right)^2 + \left(n\frac{c}{14}\right)^2}$$

So the cutoff frequencies requested are:

$$f_{11} = \sqrt{\left(\frac{c}{14}\right)^2 + \left(\frac{c}{24}\right)^2} = 24.8 \times 10^6 \text{ [Hz]} \text{ for the } TE_{11} \text{ and } TM_{11} \text{ modes.}$$

$$f_{01} = \sqrt{\left(\frac{c}{14}\right)^2} = 21.4 \times 10^6 \text{ [Hz]} \text{ for the } TE_{01}.$$

(c)

$$k_z^2 = \left(\frac{2\pi}{\lambda_g}\right)^2 = \left(\frac{2\pi f}{c}\right)^2 - \left(\frac{\pi}{12}\right)^2$$

$$\lambda_g^{-2} = \left(\frac{f}{c}\right)^2 - \left(\frac{1}{2 \times 12}\right)^2$$

$$\lambda_g = \left(\left(\frac{f}{c}\right)^2 - \left(\frac{1}{24}\right)^2\right)^{-1/2} = 36.18 \text{ [m]}$$

(d)

The tangent of the incident angle is equal to the ratio of  $k_z$  to  $k_y$ . At 15MHz, only one mode is propagating so we know that  $k_y = \frac{\pi}{12}$ .

$$k_z = \sqrt{k^2 - k_y^2} = \sqrt{\left(\frac{2\pi 15 \times 10^6}{3 \times 10^8}\right)^2 - \left(\frac{\pi}{12}\right)^2}$$

$$\frac{k_z}{k_y} = \sqrt{\left(\frac{24 \times 15 \times 10^6}{3 \times 10^8}\right)^2 - 1} = \sqrt{0.44}$$

$$\Theta_i = \arctan(\sqrt{0.44}) = 0.5857 \text{ [rad]} = 33.6^\circ$$

(e)

To get the magnetic field for the  $TM_{11}$  mode, we start with the electric field pattern for the  $TE_{11}$  mode,

replace E with H and then shift the mode spatially to match boundary conditions.

For  $TE_{11}$ :

$$E_{11}(x, y, z) = \frac{E_o}{k_o} [\hat{x}k_y \sin(k_y y) \cos(k_x x) - \hat{y}k_x \sin(k_x x) \cos(k_y y)] e^{-jk_z z}$$

So for  $TM_{11}$  we have:

$$H_{11}(x, y, z) = \frac{H_o}{k_o} [\hat{x}k_y \cos(k_y y) \sin(k_x x) - \hat{y}k_x \cos(k_x x) \sin(k_y y)] e^{-jk_z z}$$

If we assume the dimension 'a' is along x and 'b' is along y, then we can write

$$H_{11}(x, y, z) = \frac{H_o}{k_o} \left[ \hat{x} \frac{m\pi}{b} \cos\left(\frac{m\pi}{b} y\right) \sin\left(\frac{n\pi}{a} x\right) - \hat{y} \frac{n\pi}{a} \cos\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) \right] e^{-jk_z z}$$

(f)

From the expression in part e, we know that the magnetic fields are parallel to the sidewalls of the rectangular waveguide. So any currents generated must be along  $\hat{z}$ . In that case, if we were to cut thin slots along  $\hat{z}$  no currents would be interrupted and the waveguide mode would not be significantly perturbed.

(g)

To find the distance  $\delta$  we need to first find  $k_z$  for the wave.

The AM signal will penetrate furthest for the lowest mode allowed by the waveguide. From part (a) we know that this is the  $TE_{10}$  mode, with  $k_y = \pi/12$ .

$$k_z = \sqrt{(2\pi f/c)^2 - (\pi/12)^2} = j0.261$$

$$\delta = k_z^{-1} = 3.83 [m]$$

(h)

Signals in the range 88-108 MHz can propagate in several different modes. Each of these modes propagates down the tunnel at a different speed, so there will be a phase shift between the signal from the different modes. This will result in interference in the signal due to the phase difference between different modes.

## Problem 9.2

(a)

By checking boundary conditions, we see that the lowest order resonance will require two of (m,n,p) to be non-zero. Knowing the relation between frequency and angular frequency, we can rewrite equation 9.4.3 to get the frequency associated with each resonance.

$$f_{m,n,p} = \frac{\omega_{m,n,p}}{2\pi} = c \left[ \left( \frac{m}{2a} \right)^2 + \left( \frac{n}{2b} \right)^2 + \left( \frac{p}{2d} \right)^2 \right]^{0.5}$$

The lowest frequency with two of (m,n,p) non-zero will have m and p equal to one (since they are associated with the longest dimensions in this case).

$$f_{1,0,1} = c \left[ \left( \frac{1}{2 \times 0.04} \right)^2 + \left( \frac{0}{2 \times 0.03} \right)^2 + \left( \frac{1}{2 \times 0.05} \right)^2 \right]^{0.5} = 4.8 \times 10^9 \text{ [Hz]}$$

(b)

The mode associated with the lowest frequency is the  $TE_{101}$  mode. The  $TM_{101}$  mode shares the same cutoff frequency.

(c)

$TE_{101}, TE_{011}, TM_{110}$

There are two approaches to ranking the modes based on resonance frequency. First, you can put the mode numbers into the frequency equation derived in part (a) from equation 9.4.3. Second, you can recognize that lower resonance frequency is associated with longer dimensions. So the lowest frequency should have 1's associated with the two longest dimensions, the next lowest should have 1's along the longest and shortest dimensions, then the highest will have 1's along the two shortest dimensions.

(d)

We can copy this directly from equation 9.4.5.

$$\bar{E}(x, y, z) = \hat{x} E_o \sin\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi z}{d}\right)$$

$$E(x, y, z, t) = \text{Re}\{\bar{E}(x, y, z)e^{j\omega t}\} = \hat{x} \sin\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi z}{d}\right) \text{Re}\{E_o e^{j\omega t}\}$$

(e)

First we choose the time when the field has maximum value.

$$E(x, y, z, t') = \hat{x} E_{max} \sin\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi z}{d}\right)$$

$$W_e = \frac{1}{2} \varepsilon |E(x, y, z, t')|^2 = \frac{1}{2} \varepsilon E_{max}^2 \sin^2\left(\frac{\pi y}{a}\right) \sin^2\left(\frac{\pi z}{d}\right)$$

Now integrate the result over the volume.

$$w_{e,max} = \frac{b}{8} \varepsilon E_{max}^2$$

(f)

At any point, the power dissipated will be  $P_{diss}(x, y, z) = JE(x, y, z, t) = \sigma E^2(x, y, z, t)$ .

$$P_{diss}(x, y, z) = \sigma E_{max}^2 \sin^2\left(\frac{\pi y}{a}\right) \sin^2\left(\frac{\pi z}{d}\right) \cos^2(\omega t)$$

To get the total instantaneous power dissipated we need to integrate over the volume.

$$P_{diss}(x, y, z) = \frac{b}{4} \sigma E_{max}^2 \cos^2(\omega t)$$

The time average will be one half the maximum value.

$$\langle P_{diss} \rangle = \frac{b}{8} \sigma E_{max}^2$$

(g)

$$Q = \frac{\omega_{101} \langle w_t \rangle}{\langle P_{diss} \rangle} = \omega_{101} \frac{2 \frac{b}{8} \epsilon E_{max}^2}{\frac{b}{4} \sigma E_{max}^2 \cos^2(\omega t)} = \omega_{101} \frac{\epsilon_0}{\sigma} = (2\pi \times 4.8 \times 10^9) \frac{\epsilon_0}{\sigma}$$

$$\tau_{101} = \frac{Q}{\omega_{101}} = \frac{\epsilon_0}{\sigma}$$

(h)

The cavity 1/e energy decay time is the same as the electric field relaxation time.

(i)

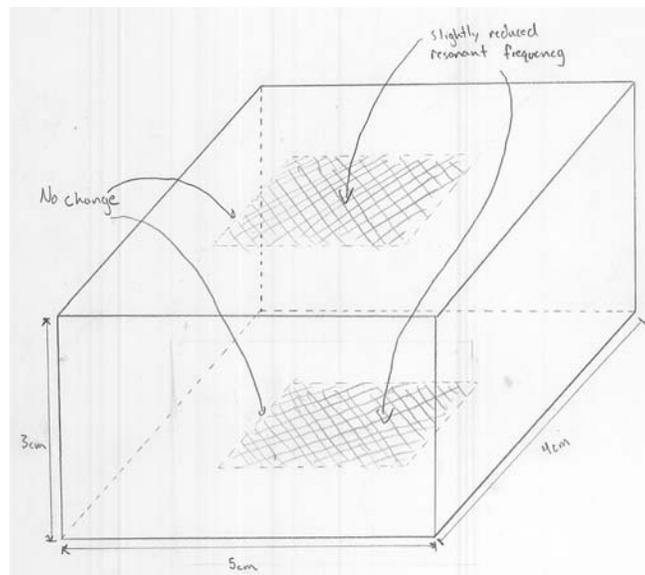
From equation 9.4.13 in the course notes we know that the change in resonance due to a perturbation of the cavity will be related to the change in stored electric and magnetic energy.

$$\frac{\Delta w_T}{w_T} = \frac{\Delta f}{f} = \Delta volume \frac{W_e - W_m}{w_T}$$

This equation tells us that if we decrease the volume slightly where the electric field is larger than the magnetic field we will reduce the resonant frequency. If we decrease the volume slightly where the electric and magnetic fields are the same then the resonant frequency will remain unchanged.

So, slightly indenting the cavity in the center of the top and bottom faces (see picture below) will reduce the resonant frequency. At the edges of the top and bottom face, the electric field goes to zero. So at some boundary between the center and edges the electric and magnetic fields will have the same value, and slightly indenting the cavity at this boundary will not change the resonant frequency.

The actual boundary will be rounder than what is shown in the picture.



**Problem 9.3**

$$G_{transmitter} = 10^{\frac{G_{db}}{10}} = 10$$

$$G_{reciever} = 10^{\frac{G_{db}}{10}} = 10$$

$$\lambda_{60MHz} = \frac{c}{f} = \frac{3 \times 10^8}{60 \times 10^6} = 5$$

**(a)**

First we need to calculate the minimum power needed to transmit the information.

$$P_{recieved,min} = 18 \times 10^6 \frac{bits}{s} \times 10^{-16} \frac{Joules}{bit} = 18 \times 10^{-10} [W]$$

$$P_{recieved,min} = P_{transmitted,min} G_{transmitter} G_{reciever} \left( \frac{\lambda}{4\pi r^2} \right)^2$$

$$P_{transmitted,min} = P_{recieved,min} \frac{1}{G_{transmitter} G_{reciever}} \left( \frac{4\pi r^2}{\lambda} \right)^2 = 18 \times 10^{-10} \frac{1}{100} \left( \frac{4\pi(60 \times 10^3)^2}{5} \right)^2 = 0.409 [W]$$

We want a margin of 20dB, so

$$20 = 10 \log_{10} \left( \frac{P_T}{P_{transmitted,min}} \right)$$

$$P_T = 100 \times P_{transmitted,min} = 40.9 [W]$$

**(b)**

$$I = P_T G_{transmitter} \frac{1}{4\pi r^2} = \frac{40.9 \times 10}{4\pi(60 \times 10^3)^2} = 9.04 \times 10^{-9} [W/m^2]$$

**(c)**

$$A_{eff} = G_{reciever} \frac{\lambda^2}{4\pi} = 10 \frac{5^2}{4\pi} = \frac{62.5}{\pi} = 19.89 [m^2]$$

**Problem 9.4**

$$G = 10$$

$$\lambda = \frac{c}{f} = 0.005 \text{ [m]}$$

(a) (Minimum Power)

Step 1: Find the intensity at the target

$$I_{ped} = P_t \frac{G}{4\pi r^2} = P_t \frac{10}{4\pi(50)^2}$$

Step 2: Find the intensity at the receiver, given the cross section of the target

$$I_{rec} = I_{ped} \sigma \frac{1}{4\pi(50)^2} = I_{ped} \frac{10^{-2}}{4\pi(50)^2}$$

Step 3: Find the power at the receiver, given the intensity at the receiver

$$P_{rec} = I_{rec} A_{eff} = I_{rec} G \frac{\lambda^2}{4\pi} = I_{rec} \frac{10\lambda^2}{4\pi}$$

Step 4: Combine

$$P_{rec} = P_t \frac{10}{4\pi(50)^2} \frac{10^{-2}}{4\pi(50)^2} \frac{10\lambda^2}{4\pi} = P_t (4\pi)^{-3} (50)^{-4} (0.005)^2$$

So the power we need to send is

$$P_t = P_{rec} (4\pi)^3 (50)^4 (0.005)^{-2} = 10^{-12} (4\pi)^3 (50)^4 (0.005)^{-2} = 0.25 (4\pi)^3 = 16\pi^3 = 496 \text{ [W]}$$

(b) (Voltage)

$$P = IV = V^2/R = V^2/100$$

$$V = \sqrt{100 \times P} = \sqrt{100 \times 496} = 223 \text{ [V]}$$

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