

## 4.1 Problem 1

### GIVEN:

A circular-cross-section pill magnet of diameter  $D$  produces a nearly uniform magnetic field,  $B_0$ , across its broad face. We wish to estimate  $B_0$  using two different methods: induced voltage and force measurement.

### 4.1.1 Part a

### FIND:

If the magnet drops through a tube, passing through with  $N$ -turn coil during the descent, how can you use the resulting voltage signal generated over the coil to estimate the magnet's field strength,  $B_0$ ?

### WORK:

Faraday's law is

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \quad (4.1)$$

In integral form, this is

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \iint \vec{B} \cdot d\vec{A} = -\frac{d\Lambda}{dt}, \quad (4.2)$$

where  $\Lambda$  is the *total* flux enclosed by the chosen path integral for  $\vec{E}$ . The path integral takes us around and around the coil  $N$  times (once for each turn; assuming that the coil is tightly wound such that all of the turns are approximately in the same place, this allows us to pull a factor of  $N$  out of the right-hand side of Eq. 4.2. In this case, we define an induced voltage (historically called the electromotive force, or  $V_{emf}$ , even though it's not a force, at all), so that

$$V_{emf} = -\frac{d\Lambda}{dt}. \quad (4.3)$$

The concept of this question is discussed in Appendix A.1.1. The math is straightforward: let  $\Lambda(t)$  be all the flux which has passed through the coil from time,  $t \rightarrow -\infty$ , to time,  $t$ . When  $t = 0$ , the magnet is exactly

halfway through the coil. Further, let  $\Phi_N$  be the total flux emanating from the north pole of the magnet, and  $\Phi_S = -\Phi_N$  the flux emanating from the south pole (negative because the flux actually enters the south pole). Then  $\Phi_N = \Lambda(t=0)/N$ , and so

$$\Phi_N = \frac{\Lambda(t=0)}{N} = \frac{1}{N} \int_{-\infty}^0 \frac{d\Lambda}{dt} dt = \int_{-\infty}^0 V dt, \quad (4.4)$$

where  $V$  is defined to be positive when the north face of the pole is above and dropping toward the coil. Applying the simple uniform field approximation to the magnet, which is accurate in the region close to the magnet pole's surface, the total flux leaving the north pole is  $\Phi_N = B_0 A$ , where  $A = \pi D^2/4$  is the cross-sectional area of the magnet and  $D$  the magnet's diameter. So,

$$B_0 = \frac{1}{NA} \int_{-\infty}^0 V dt. \quad (4.5)$$

### 4.1.2 Part b

#### FIND:

What happens when the magnet is dropped down a copper tube? Why?

#### WORK:

When you performed this experiment, you should have observed that the magnet falls very slowly through the copper tube; in fact, it rapidly reaches a terminal velocity and then descends much more slowly than it did through the plastic tube. This seems puzzling at first because the tube is copper, copper is not a ferromagnetic material (in fact, it is slightly diamagnetic, which means that it is slightly repelled in the presence of a magnetic field, and this would tend to lower friction). What's happening?

The copper tube is kind of like a stretched-out version of the inductance coil in Problem 4.1a. Each differential ring of the copper has flux passing through it, and because the field is non-uniform along the axis of symmetry, there is a time rate of change of the magnetic field enclosed by this differential loops by merit of the *motion of the magnet*. As such, an EMF is generated over these differential rings, setting up an azimuthal eddy current. There is a magnetic field associated with this current, and Lenz's Law (which may

be thought of as the sign of Faraday's Law) tells us that this current acts to oppose the change in flux; as such, the eddy currents, too, are diamagnetic and tend to repel the magnet. This slows the motion of the magnet.

The picture is a little more complicated, however. After the magnet has reached its terminal velocity, it must be that the force from the eddy currents exactly balances the force from the gravitational field. So the kinetic energy of the magnet is no longer changing, and nor is the energy stored in the magnetic field. But the potential energy must be changing because the magnet is still falling through the gravity field. If the pill is not accelerating, where's the energy going? At this point, the usual 6.013 approximation that copper is a perfect conductor fails us; if it were, then the energy could not be dissipated anywhere. As such, it must be that the losses in the copper tube walls are important. In fact, the steady state is reached when the power dissipated in the copper tube walls exactly balances the time rate of change of potential energy, which is  $mgv$ , where  $m$  is the mass of the magnet,  $g$  is the acceleration due to gravity, and  $v$  is the magnet's terminal velocity. The terminal velocity is slowest when the resistive "load" in the copper wall is "matched" to the inductive source in the form of the falling magnet. And there is no terminal velocity due to magnetic effects in the limits of zero or infinite conductivity (though the case of infinite conductivity may make getting the magnet into the tube tricky).

### 4.1.3 Part c

#### **FIND:**

Using the breakaway force from pulling the magnet away from a high- $\mu$  plate, again estimate the field strength,  $B_0$ , of the magnet in the uniform-field approximation.

#### **WORK:**

Approximating the magnet's field as uniform simplifies this problem considerably. In this case, we can quickly estimate the gap energy density as  $W_m = B_0^2/2\mu_0$ , and the force as  $F = W_m A = AB_0^2/2\mu_0$ . This means that the field strength of the magnet is  $B_0 = \sqrt{\frac{2\mu_0 F}{A}}$ . In your experiment, the scale's readout of mass assumes (correctly) a downward force from gravity; you determine how much of the gravitational load the magnetic attraction

can support. Therefore, it is appropriate to determine the breakaway force from  $\Delta mg$ , where  $\Delta m$  is the mass differential you read off from the digital. This gives for the magnet's field strength

$$B_0 = \sqrt{\frac{2\mu_0\Delta mg}{A}}, \quad (4.6)$$

where again,  $A = \pi D^2/4$  is the cross-sectional area of the magnet.

The uniform field approximation is good when the magnet is close to and aligned with the high-permeability material because the field lines are straightened out such that they extend along normal lines from the magnetic surface to the ferromagnetic plate's surface.

## 4.2 Problem 2

### GIVEN:

Linear reluctance motor energized by N-turn coil; stator and linear shaft have permeability,  $\mu \gg \mu_0$ . At  $t = 0$ , overlap of shaft and stator is area,  $A = 9 \times 10^4 \text{ [m}^2\text{]}$ . Depth of system is  $3\text{cm}$ .

### 4.2.1 Part a

### FIND:

Maximum magnetic energy density  $[\text{J/m}^3]$  at  $t = 0$ , as well as location of maximum energy density.

### WORK:

The maximum energy density is located in the gap. You can satisfy yourself of this by considering the field distribution. The high- $\mu$  of the stator core confines the magnetic field lines largely to the interior of the core (this is the lowest-energy configuration). To complete their (magnetic) circuit, these field lines must eventually exit the stator, since they cannot take a closed-circuit path. They will do this primarily where their path through the air is a minimum - this is in the air gap between the linear shaft and the stator. Since total flux is conserved, the simplest approximation is to neglect fringing fields and assume that all of the flux passes through the smallest air gaps.

$H$  in the gap is found most simply from Ampère's law in the approximation that since  $B_{\text{in stator}} \sim B_{\text{gap}}$  (depending on on the cross-section-to-gap area ratio), but  $B = \mu H$  and  $\mu_{\text{in}} \gg \mu_0$  such that  $H$  is negligible in the Ampèrian countour everywhere except in the narrow air gap. Then

$$\oint \vec{H} \cdot d\vec{l} \approx H2g = I_{\text{enc}} = NI \quad (4.7)$$

where  $g$  is the gap length both above and below the shaft, so that  $H = \frac{NI}{2g}$ .

The energy density in the gap is found directly from

$$\boxed{W_m = \frac{1}{2}\mu_0 H^2 = \frac{1}{2}\mu_0 \left(\frac{NI}{2g}\right)^2}, \quad (4.8)$$

so that the total energy is this density multiplied by the volume of the gap,  $V = 2gA$ ,

$$w_m = \frac{1}{2}\mu_0 H^2 V = \frac{1}{2}\mu_0 \left(\frac{NI}{2g}\right)^2 2gA = \frac{1}{2}\mu_0 \frac{(NI)^2 A}{2g}, \quad (4.9)$$

For a look at how this problem might be approached from the perspective of magnetic circuits, please take a look at Appendix A.2.1.

## 4.2.2 Part b

### FIND:

Force,  $f_z$ , acting to pull the pole faces together.

### WORK:

Appendix A.2.2 includes a discussion explaining how to think about the force calculations shown here (including how to get the right sign). And to make sense of where the flux,  $\lambda$  comes in, see A.2.1. In the meantime, define  $\mathcal{R} = 2g/(\mu_0 A)$  and let  $\lambda$  be the total flux enclosed by the coil; then it can be shown that

$$\vec{f} = -\nabla w_m = -\nabla\left(\frac{1}{2}\lambda^2\mathcal{R}\right), \quad (4.10)$$

so that when the flux,  $\lambda$ , is held constant (since the system is being controlled by a voltage source rather than current),

$$\vec{f} = -\frac{1}{2}\lambda^2\nabla\mathcal{R}. \quad (4.11)$$

In the  $\hat{z}$ -direction, this is

$$f_z = -\frac{1}{2}\lambda^2\frac{\partial\mathcal{R}}{\partial z}. \quad (4.12)$$

Remember that  $\mathcal{R} = 2g/(\mu_0 A)$ , where the  $2g$  extends in the  $z$ -direction. By symmetry, if the stator is squeezed or expanded an amount,  $dz$ , the gap lengths change by  $dg = dz/2$ , so that  $d/dz \rightarrow \frac{1}{2}d/dg$ . Finally,  $d\mathcal{R}/dz = d\frac{1}{2}\mathcal{R}/dg = 1/(\mu_0 A)$ , so that

$$f_z = -\frac{1}{2}\lambda^2 \frac{\partial \mathcal{R}}{\partial z} = -\frac{1}{2} \frac{\lambda^2}{\mu_0 A} = -\frac{1}{2} (BA)^2 / (\mu_0 A) = -\frac{B^2}{2\mu_0} A = -W_m A. \quad (4.13)$$

And there it is: the force is the magnetic stress times the normal area, which we knew all along. To evaluate, we know that  $B$  in the gaps is 2 T; therefore,  $f_z = -(2[\text{T}])^2 / (4\pi \times 10^{-7} [\text{H/m}]) 9 \times 10^{-4} [\text{m}^2] = -1.432 \times 10^3 [\text{N}]$ , where the negative sign indicates that the force acts to oppose an increase in the gap (i.e. it pulls the pole faces toward one another).

In terms of natural problem parameters, the force might also be expressed as

$$f_z = -\frac{1}{2\mu_0 A} \left( \frac{NI}{\mathcal{R}} \right)^2 = -\frac{1}{2\mu_0 A} \left( \frac{NI\mu_0 A}{\ell} \right)^2 = -\frac{\mu_0 A}{2} \left( \frac{NI}{2g} \right)^2 = -\frac{\mu_0 A (NI)^2}{8g^2} \quad (4.14)$$

Don't worry - the negative sign is just what we wanted! Since we did this carefully, we assumed a differential motion in the  $\hat{z}$ -direction which is implicitly in the *positive*  $\hat{z}$ -direction. The sign on the force then naturally points to oppose this differential motion, just as we expected. The force results in a stress on both the cantilevered stator and the shaft; mechanical engineers need to know these stresses in order to properly design the components of electric machines like this one.

### 4.2.3 Part c

**FIND:**

Lateral force,  $f_x$ , pulling the sliding member into the gap.

**WORK:**

The answer will be  $f_x = \frac{B^2}{2\mu_0} D 2g = 9.549 [\text{N}]$ . Let's do this the hard way for learning, but remember the not-so-hard-way for doing quickly. Hard way: with  $w_m$  = total magnetic energy =  $\frac{1}{2}\lambda^2 \mathcal{R}$ ,

$$f_x = -\frac{\partial w_m}{\partial x} = -\frac{\lambda^2}{2} \frac{\partial}{\partial x} \left( \frac{\ell}{\mu_0 A} \right) = \frac{\lambda^2 \ell}{2\mu_0 A^2} \frac{\partial A}{\partial x} = \frac{\lambda^2 \ell D}{2\mu_0 A^2} = \frac{B^2}{2\mu_0} D 2g, \quad (4.15)$$

where  $D$  is the depth of the unit and the overlap area is  $A = xD$ , and again,  $\ell = 2g$  is the total gap length. This is just what we expected. Note that the lateral force on the shaft points in the direction to *increase* the overlap area - the shaft is pulled into the stator (or equivalently, the stator is pulled toward the shaft).

#### 4.2.4 Part d

**FIND:**

Output voltage induced over the  $N$ -turn coil if the shaft is withdrawn at a velocity,  $v$ , assuming the instantaneous overlap area is still  $9 \times 10^{-4}[\text{m}^2]$ .

**WORK:**

The magnetic circuit gives the simple Ohmic-styled relation,  $NI = \lambda\mathcal{R}$ , so that  $\lambda = NI/\mathcal{R} = NI\mu_0 A/\ell$ . Faraday's law then asserts that the induced emf voltage is the time rate of change of the total enclosed flux,  $V_{emf} = -d\lambda/dt$ ; the sign of the circuit voltage depends on the sense in which the leads are attached. Given that the problem statement allows for a varying voltage over the leads, let us assume that now, the current-turn product is clamped by a current source, so that  $NI$  is a constant. Then

$$\boxed{V = -N \frac{d}{dt} \left( \frac{NI\mu_0 A}{\ell} \right) = -N \frac{NI\mu_0 D}{\ell} \frac{dx}{dt} = NB_g Dv = 0.06Nv[\text{V}]} \quad (4.16)$$

The voltage is positive because withdrawing the rod tends to reduce the flux passing through the magnetic circuit, and Lenz's law tells us that the induced current is set up in such a way as to oppose this change in flux. This means that the current will try to send more flux through the magnetic circuit in the original direction, which requires an induced voltage that strengthens the original current.

## 4.3 Problem 3

### GIVEN:

Leaky capacitor, everywhere permittivity,  $\epsilon$ , but having a conductivity,  $\sigma_a$  from the top electrode to the mid-plane, and  $\sigma_b$  from the mid-plane to the bottom electrode.  $\sigma_a = 10\sigma_b$ . Total dielectric thickness is  $2d$ . The capacitor is charged to  $1000 V_{DC}$ .

### 4.3.1 Part a

### FIND:

Total capacitance of this device.

### WORK:

Neglecting fringing fields, the geometry of this problem lends itself to the assumption that all field quantities are uniform in the  $\hat{x}$ - and  $\hat{y}$ -directions (where the  $\hat{z}$ -direction points from the bottom electrode to the top electrode). As such, the equipotential lines should be planes parallel with the bottom and top electrodes. By continuity of the potential, it must be that there exists an equipotential in the interfacial plane between the two dielectrics; that is, everywhere on that plane is of uniform potential. This allows us to break up this inhomogeneous problem into two homogeneous problems that we already know how to solve; the boundary condition at the interface knits the two solutions together. In particular, we can treat the top half of the system as a leaky capacitor, and the bottom half as another leaky capacitor (*leaky capacitor* refers to the fact that the structure has both capacitive and resistive character - it “leaks” charge in DC).

So, the capacitance of the top half is  $C_a = \epsilon A/d$ , and that of the bottom half is the same,  $C_b = \epsilon A/d = C_a$ . The resistance of the top half is  $R_a = d/(\sigma_a A)$ ; of the bottom half,  $R_b = d/(\sigma_b A) = 10R_a$ . An equivalent circuit can be drawn as in Figure 4.1.

In the *high-frequency limit*, the resistances do not contribute significantly to the total impedance of the structure, so the circuit is reduced to two series capacitances. These combine like resistors in parallel:  $C_{total} = C_1 C_2 / (C_1 + C_2)$ , so

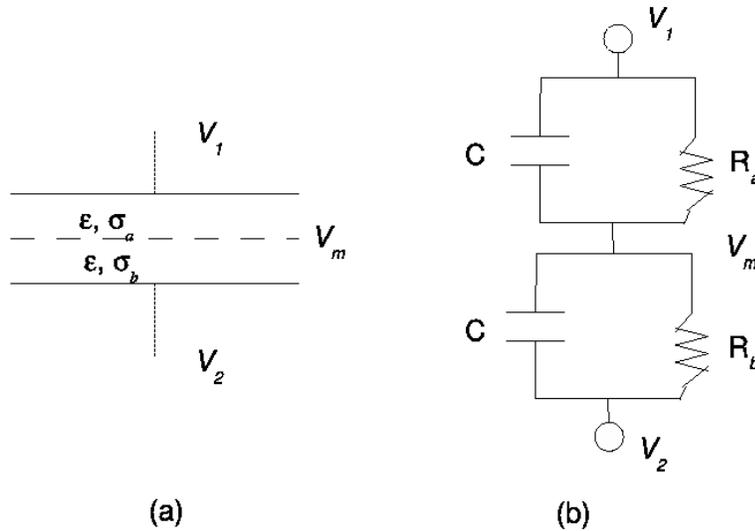


Figure 4.1: (a) Sketch of the actual problem geometry; (b) equivalent circuit of the problem.

$$\boxed{C_{total} = \frac{C_a C_b}{C_a + C_b} = \frac{C_a}{2} = \frac{\epsilon A}{2d}}, \quad (4.17)$$

which is just the capacitance of the structure excluding the inhomogeneity. This makes sense because in the high-frequency limit, the effective impedance for the displacement current vanishes, so that the displacement current (the second half of Ampère's law) shorts out the ohmic current (the first part). In this case, conductivity is no longer important, and the inhomogeneity in the dielectric effectively disappears.

### 4.3.2 Part b

**FIND:**

Voltage,  $V_m$ , at the midpoint junction between the two dielectrics.

**WORK:**

The simplest way to solve this problem is to use the circuit diagram in Figure 4.1.  $V_m$  is the node voltage in between the two parallel RC pairs. Since the capacitor is maintained at a DC voltage, it is now appropriate to look at the circuit in the *low-frequency* limit, where the capacitors are open circuits. In this case, the resistors comprise a *voltage divider*, so that  $V_m = VR_b/(R_a + R_b)$ . With  $R_b = 10R_a$  (see Part a),  $V_m = 10V/11 = 1000[\text{V}]/1.1 \approx 909[\text{V}]$ .

**4.3.3 Part c****FIND:**

Net free and polarization surface charge densities,  $\rho_{fm}$  and  $\rho_{pm}$ .

**WORK:**

Perhaps the field approach provides the simplest method for finding the charge on the interfacial layer. The program will be as follows:

1. find the fields in each homogeneous region
2. apply the boundary condition for the component of  $\vec{D}$  normal to the boundary and extract the surface charge density from the difference in  $D_n$ .

Neglecting fringing fields, the  $\vec{E}$  field in region  $a$  is  $\vec{E}_a = -\hat{z}(V - V_m)/d$ , while in region  $b$ ,  $\vec{E}_b = -\hat{z}(V_m)/d$ . With the constitutive relation,  $D = \epsilon E$ ,  $D_a = \epsilon_a E_a = -\hat{z}\epsilon(V - V_m)/d$  and  $D_b = -\hat{z}\epsilon(V_m)/d$ . The  $\hat{z}$  components are normal to the interface, so

$$\rho_{fm} = D_{bz} - D_{az} = \frac{\epsilon}{d}(V_m - (V - V_m)) = \frac{\epsilon}{d}(2V_m - V), \quad (4.18)$$

or

$$\rho_{fm} = \frac{\epsilon}{d}V \frac{R_b - R_a}{R_b + R_a} = \frac{\epsilon}{d}V \frac{\sigma_a/\sigma_b - 1}{\sigma_a/\sigma_b + 1} = \frac{9000}{11} \frac{\epsilon}{d} [\text{C/m}^2] \approx 818.2 \frac{\epsilon}{d} [\text{C/m}^2]. \quad (4.19)$$

A polarization surface charge also exists. To determine it, start with

$$\nabla \cdot \vec{P} = -\rho_p, \quad (4.20)$$

where the negative sign for the *volumetric polarization density*,  $\rho_p$ , indicates that the polarization field lines extend from negative to positive polarization charges. Performing the same pillbox integration trick that led to the boundary condition for  $D$  normal to the boundary will give

$$P_{n2} - P_{n1} = -\rho_{ps}, \quad (4.21)$$

where the polarization lines point from medium 1 to medium 2 and  $\rho_{ps}$  is the polarization surface charge. By definition,  $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ , so  $\vec{P} = \vec{D} - \epsilon_0 \vec{E}$ . Then the normal component pops out as  $P_n = D_n - \epsilon_0 E_n$ . Then

$$\boxed{\rho_{pm} = P_{na} - P_{nb} = D_{na} - D_{nb} - \epsilon_0(E_{na} - E_{nb}) = (E_{nb} - E_{na})(\epsilon_0 - \epsilon)} \quad (4.22)$$

or

$$\boxed{\rho_{pm} = V \frac{\epsilon_0 - \epsilon}{d} \frac{\sigma_a/\sigma_b - 1}{\sigma_a/\sigma_b + 1} \approx 818.2 \frac{\epsilon_0 - \epsilon}{d} [\text{C/m}^2]} \quad (4.23)$$

This is expected to be a negative quantity.

#### 4.3.4 Part d

##### FIND:

If the capacitor is shorted momentarily such that the charge at plane,  $m$ , remains constant, before being opened again, to what peak voltage,  $V_p$ , does the external voltage rise before decaying to zero? With what approximate time constant does the open circuit voltage decay? Does this present a dangerous situation?

##### WORK:

Figure 4.1 again provides a simple means for solving this problem, though the field approach informs us what happens physically and helps to define initial conditions and predict steady-state behavior.

At time  $t = 0^-$ , the circuit is shorted, but the charge at the interface,  $m$ , is unable to discharge. At time  $t = 0$ , the circuit is then opened. Gauss' law

prescribes the fields in both regions,  $a$  and  $b$ :  $(D_a + D_b)A = Q_m \Rightarrow \epsilon(E_a + E_b) = \rho_{fm} = 2V_{m0}\epsilon/d$ , since  $E_a = E_b = (V_m - V_{top})/d = (V_m - V_{bottom})/d$  and  $V_{top} = V_{bottom}$  at time  $t = 0$ , and defining  $V_{bottom}$  to be the ground. This means that the initial conditions are:

$$V_m(t=0) = \frac{\rho_{fm} d}{2 \epsilon} = \frac{V_0 R_b - R_a}{2 R_b + R_a} = \frac{9000}{22} [\text{V}] \approx 409.1 [\text{V}] \quad (4.24)$$

$$V_1(t=0) = 0$$

where  $V_1(t)$  is the voltage on the top electrode,  $V_m(t)$  is the voltage at the midplane interface, and both of these voltages are referenced to the bottom electrode. Next, we can write the differential equations that govern these voltages:

$$i_{R_a} = i_{C_a} = (V_m - V_1)/R_a = -V_{C_a}/R_a = C_a \frac{dV_{C_a}}{dt}$$

$$\Rightarrow \frac{dV_m - V_1}{dt} = -\frac{1}{R_a C_a} (V_m - V_1)$$

$$\frac{dV_m}{dt} = -\frac{V_m}{R_b C_b}$$

These are two decoupled differential equations which may be solved immediately as

$$(V_m - V_1) = (V_m - V_1)_{t=0} e^{-t/\tau_1} V_m = V_m(t=0) e^{-t/\tau_2} \quad (4.25)$$

where  $\tau_1 \equiv R_a C_a$  and  $\tau_2 \equiv R_b C_b$ . From this follows (noting that  $(V_m - V_1)_{t=0} = V_m(t=0)$  since the top and bottom electrodes are initially shorted and the bottom electrode is always the reference)

$$V_1 = V_m(t) + V_{m0} e^{-t/\tau_1} = V_{m0} (e^{-t/\tau_2} - e^{-t/\tau_1}) \quad (4.26)$$

$V_1$  is the required voltage; it is the potential difference between the top and bottom electrodes after the system has been shorted for a moment and then opened. Since  $\sigma_a = 10\sigma_b$  but all other parameters of the two homogeneous regions are identical,  $R_b = 10R_a$  and  $\tau_2 = 10\tau_1$ . This means that while the two exponentials initially balance to zero, the negative term in the equation decays rapidly, leaving behind a positive excess that decays more slowly as approximately  $\tau_2 = \epsilon/\sigma_b$ . The peak voltage is found by locating the critical point for this equation where the derivative with respect to time is zero. This is  $t^* = \ln(\tau_2/\tau_1)\tau_1\tau_2/(\tau_2 - \tau_1)$ . Substituting into Equation 4.26 gives for the peak voltage

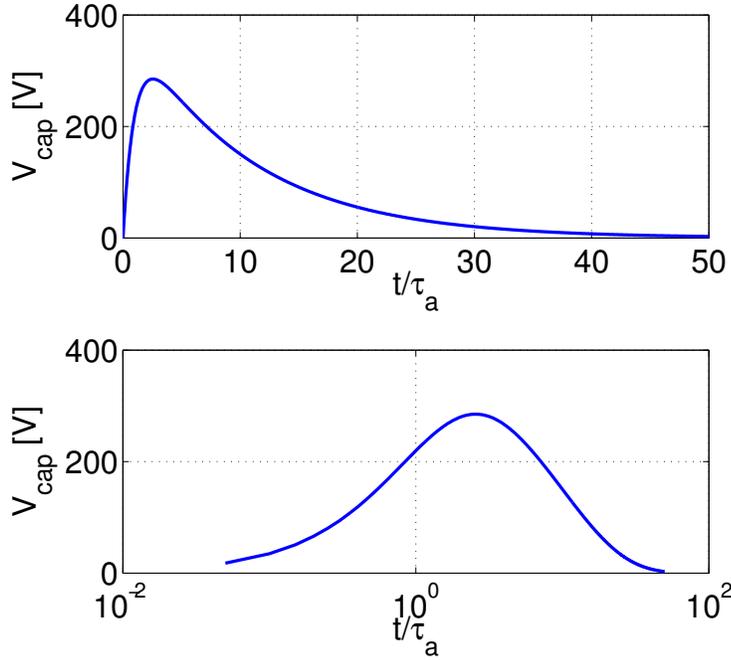


Figure 4.2: Voltage over the capacitor; time is non-dimensionalized with  $\tau_a = \epsilon/\sigma_a$ . Initially, the voltage grows as  $(1-e^{-t/\tau})$  (not exponential growth, but exponential asymptote), and finally decays with time constant,  $\tau_b = \epsilon/\sigma_b = 10\tau_a$ . The top figure is in linear time; the bottom plots time on a logarithmic axis.

$$\begin{aligned}
 V_p &= V_{m0} \left[ \left( \frac{\tau_2}{\tau_1} \right)^{-\tau_1/(\tau_2-\tau_1)} - \left( \frac{\tau_2}{\tau_1} \right)^{-\tau_2/(\tau_2-\tau_1)} \right] \\
 &= V_{m0} (10^{-1/9} - 10^{10/9}) \approx 0.6968 V_{m0} \approx 285.1 [\text{V}]
 \end{aligned}
 \tag{4.27}$$

Figure 4.3.4 shows the voltage over the whole capacitor over time.

In summary, the voltage over the capacitor is shorted for a moment, then creeps up to over a quarter of its original value and decays with a time constant of approximately  $\tau_2 = \epsilon/\sigma_b$ . This may not be surprising to us

because we knew that the behavior of this system is akin to two RC pairs in series. *But* this is not what the problem looks like in the lab! Remember that we have been modeling a single high-voltage capacitor in a single package. A technician who thinks he or she is playing it safe by discharging the capacitor find find himself or herself in for a nasty shock if the discharge was too quick and the handling afterward too soon.

## 4.4 Problem 4

### GIVEN:

Transistor controlled by free charge density,  $\rho_f$ , within a zone of width,  $W$ , where  $\epsilon = 4\epsilon_0$  and  $\sigma = 1$  [S/m].

### FIND:

If the voltages on the transistor electrodes bounding that zone abruptly change, forcing  $\rho_f$  to take values characterizing the next transistor state, with what time constant,  $\tau$ , is the new  $\rho_f$  distribution established? Can  $\tau$  be significantly less than  $W/c$ , where  $c$  is the speed of light within the semiconductor? Discuss briefly.

### WORK:

Assuming a medium is linear, homogeneous, and isotropic, and also has ohmic and dielectric character, the time constant for charge decay is  $\tau = \epsilon/\sigma$ . This follows immediately from taking the divergence of Ampère's law, substituting from Gauss' law for electric fields, and solving the ordinary differential equation for the free charge density,  $\rho_f$ .

Let's examine the quantity numerically with the given parameters:  $\tau = \epsilon/\sigma = 4\epsilon_0/\sigma \approx 3.54 \times 10^{-10}$  [s]. For comparison, the propagation time for information to be transferred between the electrodes is  $W/c$ . Dimensions of integrated circuits are already in the tens of nanometers. Conservatively assuming  $W \sim 100$  [nm], and noting that when  $\epsilon = 4\epsilon_0$ ,  $c = c_0/2$  in a nonmagnetic material, this leads to  $W/c \sim 10^{-7}/(3 \times 10^8/2) = 2 \times 10^{-15}/3$  [s]. In this example, the light propagation time is much smaller than the exponential decay rate of the charge density; however, conductivity varies over many orders of magnitude, and so this need not always be the case.

The question of whether the exponential decay time can exceed the propagation time of information through the structure is subtle. However, the basic idea is that the speed of light is the limiting speed of *information* and *mass* transfer. The case of changing  $\rho$  would seem like *mass transfer* because we know that the charges are attached to masses which are certainly moving inside the structure. However, the actual quantity is *density*. Consider a discretized grid in 2D. Imagine four particles are located in a square in this grid, and that these four particles each move along separate diagonals into

the corner boxes of a  $9 \times 9$  subgrid. In the region occupied by particles, the density is transitioned from 4 particles/grid to  $4/9$  particles per grid, while the particles have only moved a distance of  $\sqrt{2}$  box lengths. This example serves to show that the density can vary rapidly even if the particles vary slowly.

However, it is true that ohm's law is limited in that it describes the steady-state, average motion of particles that is set up after many collisions. First of all, this average velocity must always be smaller than that of light, and second, the transport cannot be considered "ohmic" until many collisions have occurred (i.e. until after many collision times,  $\tau_c$ ). Before this time, the charged particles behave more like free charged particles accelerating continuously under an electric field. So, we can say with a fair degree of certainty that a charged particle cannot hope to travel from one electrode to another in a time shorter than  $W/c$  (and if it did, physicists would be very excited and would have something to do).

A simpler problem to consider that might illustrate this point is that of a regular parallel plate capacitor. Assume a charge carrier density of  $n$  charge carriers per unit volume, and that each charge carrier has charge,  $q$ . Also, imagine that the capacitor has area,  $A$ , thickness,  $d$ , dielectric constant  $\epsilon$ , and conductivity,  $\sigma$  (i.e. it is a "leaky" capacitor). If the capacitor is charged up initially to  $V_0$  and then open-circuited, it will discharge itself according to  $V = V_0 e^{-t/\tau}$ , where  $\tau = \epsilon/\sigma$ . This implies a current,  $i = V/R = V\sigma A/d = V_0 e^{-t/\tau}/R$ . The current density is  $J = i/A$ , and the corresponding velocity of charge carriers,  $v$ , must then be  $J = nqv \Rightarrow v = J/(nq) = i/(nqA) = V_0 e^{-t/\tau}/(RnqA) = V_0 e^{-t/\tau} \sigma/(dnq)$ .

We run into trouble when  $v > c$ , the speed of light. In this case, we had better check our assumptions - the two constitutive relations,  $D = \epsilon E$  and  $J = \sigma E$ . It is the second of these - the point form of Ohm's law - that is failing - it predicts that the charge carriers can go faster than they actually can.

Again, let's pick some typical numbers. For a voltage of 1 V,  $\sigma = 1$  [S/m],  $d=100$  [nm],  $n = 10^{16}$  [m<sup>-3</sup>], and  $q = 1.6 \times 10^{-19}$  [C], the maximum velocity predicted using Ohm's law is  $6.25 \times 10^9$  [m/s], over an order of magnitude faster than the speed of light. It would seem we had better be careful before assuming that ohmic transport is the dominant transport mechanism in semiconductors for these time and space scales.

## 4.5 Problem 5

### GIVEN:

Plane wave with 1 GHz frequency and power flux density, 1 kW/m<sup>2</sup> incident on a black absorbing surface.

### FIND:

Average pressure,  $p$ , exerted on surface by plane-wave radiation.

### WORK:

There are several ways to approach this problem. The direct method from electromagnetics is as follows: the instantaneous momentum of a plane wave,  $\vec{q}$ , is

$$\vec{q}/A = \frac{\vec{S}}{c^2} = \frac{\vec{E} \times \vec{H}}{c^2}, \quad (4.28)$$

where  $\vec{S}$  is the Poynting vector and  $c$  is the speed of light. The average momentum flux traversing a unit area per unit time is then

$$\left\langle \frac{d\vec{q}}{dt} \right\rangle / A = c \frac{\langle \vec{S} \rangle}{c^2} = \hat{s} \frac{1}{2\eta} \frac{E^2}{c} = \frac{\epsilon E^2}{2}, \quad (4.29)$$

where  $\hat{s}$  is the unit vector pointing in the direction of the wave's propagation. The extra factor of  $c$  appears because we compute the average momentum contained by the wave and then examine the momentum flux (i.e. how much of that momentum is crossing a reference plane per unit area per unit time).

In this case, the surface is black, so all the momentum flux striking it is absorbed. The pressure is  $p = F/A = dq/dt/A = \epsilon E^2/2$ . It is given that  $\langle S \rangle = 1$  [kW/m<sup>2</sup>]; hence  $p = \langle S \rangle / c \approx 10^3 [\text{W/m}^2] / 3 \times 10^8 [\text{m/s}] \approx 3.333 \times 10^{-6} [\text{N/m}^2]$ .

Another approach is to consider the equivalent flux of photons. The photon energy is  $hf$ , and to achieve a power flux of 1 [kW/m<sup>2</sup>], there must be  $\langle S \rangle = hfn c \Rightarrow n = \langle S \rangle / (hfc)$  photons per unit volume traveling at the speed of light. The momentum of each photon is  $q = h/\lambda$ ; as such, the momentum flux (which is also the pressure) is

$$p = \frac{h}{\lambda} n c = \frac{h}{\lambda} \frac{\langle S \rangle}{hfc} c = \frac{\langle S \rangle}{f\lambda} = \frac{\langle S \rangle}{c} \approx 3.333 \times 10^{-6} [\text{N/m}^2], \text{ as before.}$$

## 4.6 Problem 6

### GIVEN:

Laplace's equation for  $\Phi$  in 3D; boundary conditions are such that separation constant in each dimension is 0.

### 4.6.1 Part a

### FIND:

General form of solution for  $\Phi$  for this case.

### WORK:

Laplace's equation is

$$\nabla^2\Phi = 0. \quad (4.30)$$

In Cartesian coordinates, this is

$$\nabla^2\Phi = \partial_x^2\Phi + \partial_y^2\Phi + \partial_z^2\Phi = 0, \quad (4.31)$$

introducing the notation,  $\partial_{x_i} \leftrightarrow \frac{\partial}{\partial x_i}$ . Assuming the solution is separable,  $\Phi(x, y, z) = X(x)Y(y)Z(z)$ , and substituting into Laplace's equation gives

$$\nabla^2\Phi = YZ\partial_x^2X + XZ\partial_y^2Y + XY\partial_z^2Z = 0, \quad (4.32)$$

from which it follows that

$$\frac{\partial_x^2X}{X} + \frac{\partial_y^2Y}{Y} + \frac{\partial_z^2Z}{Z} = 0. \quad (4.33)$$

The only way this can be true for arbitrary  $x$ ,  $y$ , and  $z$  is if each ratio term is a constant. Let  $k_x^2 = (\partial_x^2X)/X$ ,  $k_y^2 = (\partial_y^2Y)/Y$ , and  $k_z^2 = (\partial_z^2Z)/Z$ ; then  $k_x^2 = -(k_y^2 + k_z^2)$ . For this problem, we will assume that the boundary conditions are such that  $k_x^2 = k_y^2 = k_z^2 = 0$ . Integrating *twice* each decoupled component,  $X(x)$ ,  $Y(y)$ , and  $Z(z)$ , separately yields  $X(x) = Ax + B$ ,  $Y(y) = Cy + D$ , and  $Z(z) = Ez + F$ . Now, multiply all together to get the most general solution for these separation constants:

$$\begin{aligned}\Phi(x, y, z) &= X(x)Y(y)Z(z) = (Ax + B)(Cy + D)(Ez + F) \\ &= C_1xyz + C_2xy + C_3yz + C_4zx + C_5x + C_6y + C_7z + C_8,\end{aligned}\tag{4.34}$$

where the constants,  $C_i$ ,  $i$  from 1 to 8, are chosen to satisfy the boundary conditions.

### 4.6.2 Part b

#### FIND:

If the potential is  $\Phi = 0$  on three intersecting edges of a cube, and 10 [V] at the opposite corner, what is the value of  $\Phi$  at the intermediate corners? (Note: Laplace's equation is ubiquitous; we could as well be solving for the temperature at these corners, for example.)

#### WORK:

The general solution in Equation 4.34 may be applied to satisfy the given boundary conditions. Place the coordinate system along the vertex where the three zeroed edges intersect. Along any axis, at least two coordinates are zero, this means that the first four constants in the general solution can be non-zero, but all the rest must be zero. But symmetry also restricts the solution. For the expanded form, you can readily convince yourself that  $C_2 = C_3 = C_4 = C$ . For the factored form, in order to require that  $C_5$ ,  $C_6$ , and  $C_7$  are zero, we need that  $DF = BF = BD = 0$ . But symmetry in this case requires that the problem is invariant if the axes are permuted; as such,  $A = C = E$  and  $B = D = F$ . In other words, symmetry means that Equation 4.34 can be written as

$$\Phi(x, y, z) = (Ax + B)(Ay + B)(Az + B)\tag{4.35}$$

Then it follows that  $B = D = F = 0$ . Immediately, we recognize that the only term that survives is the  $xyz$  term. If the cube has length,  $\ell$ , then  $10 = C_1\ell^3 \Rightarrow C_1 = 10/\ell^3$  ([V] in this application). Along the intermediate corners, either  $x$  or  $y$  or  $z$  is zero; therefore, the  $xyz$  term vanishes, leaving  $\Phi(\text{intermediate corners}) = 0$ .

# Appendix A

## Appendix for HW4

### A.1 Problem 1

#### A.1.1 Part a: Concept

The important conceptual point of this question is that the induced voltage signal records all of the flux which passes through it (scaled by a factor of  $N$ ). Since the magnet passes through the coil from north to south pole, and the field lines all emanate from the north pole and enter at the south pole, *all* of the flux passing out of the north pole of the magnet will pass through the coil, and *afterward*, *all* of flux entering the south pole will pass through the coil. But Faraday's law in Eq. 4.3 tells us that the incremental change in flux passing through the coil per unit time is exactly the induced voltage (with a factor of  $N$ ). This means that the voltage keeps track of how much new flux is passing through the coil at any given instant. If we add up (integrate) all of these incremental changes for the entire descent, we'll get zero, because about half of the time, the flux passes through the coil in one direction, and the other half of the time, it passes through in the other direction. But if we just integrate the incremental flux changes up until the time the magnet is halfway through the coil, we must have found how much flux is leaving the northern face of the magnet.

This is an incredible result, because it allows us to have a very fundamental characterization of the magnet without knowing very much about the exact nature of the fields or about the magnet's motion. And we can even use this flux to extract a  $B_0$  for our simple model of the fields from the magnet, even though this model could not have produced the voltage trace

in the manner prescribed in the problem statement.

## A.2 Problem 2

### A.2.1 Part a: Magnetic Circuits

Let's calculate the magnetic flux,  $\lambda$ , in the system. We know  $H$  in the gap, and so we can find  $B$  here. In the uniform field approximation,  $\lambda_{gap} = B_{gap}A = \mu_0 H_{gap}A = NI \frac{\mu_0 A}{\ell}$ , where  $\ell = 2g$  is the total path length through the air gaps. By conservation of flux, and in the approximation that all the flux crosses through air only at the narrowest gaps, we know that  $\lambda_{gap} = \lambda_{stator} = \lambda_{shaft}$  - that is, all the flux exiting the  $NI$  coil completes a circuit through the stator and shaft. This should turn on a light in your memory.

Let's examine this relation a little more closely: how about putting  $NI$  on one side, so that we have

$$NI = \frac{\ell}{\mu_0 A} \lambda. \quad (\text{A.1})$$

This relation should set off another light. The first term looks suspiciously like the resistance in a straight block of ohmic material, where the conductivity,  $\sigma$ , is replaced by the permeability,  $\mu_0$ . And the rest of the relation looks like Ohm's law with the substitution,  $NI \leftrightarrow V$  and  $\lambda \leftrightarrow I$ . In fact, the analogy between electrostatic circuits and *magnetostatic* circuits is complete. We define the quantity,  $\mathcal{R} = \frac{\ell}{\mu_0 A}$ , as the magnetic *reluctance*, analogous to electrical resistance. The current-turn product,  $NI$ , is sometimes referred to as the *magnetostatic force*, or *MMF*, even though (again) it's not a force at all.

In general, we can break up the Ampèrian contour integral into the sum of  $H \cdot \ell$  products in each homogeneous sub-medium, and then impose conservation of flux, so that

$$NI = \oint \vec{H} \cdot d\vec{l} \approx \sum_{i=1}^n H_i \ell_i = \sum_{i=1}^n \frac{B_i}{\mu_i} \ell_i = \sum_{i=1}^n \frac{\ell_i}{\mu_i A} \lambda_i = \sum_{i=1}^n \mathcal{R}_i \lambda_i \quad (\text{A.2})$$

for any closed path with  $n$  homogeneous regions. This is Kirchoff's loop law for magnetostatic circuits. Kirchoff's node law is the statement that all of the flux entering a "node" also exits the node (conservation of magnetic flux).

Why is this useful? Because it's now relatively simple to calculate the flux,  $\lambda$ , the inductance,  $L = \lambda/i$  or mutual inductances for more complicated systems, and other useful quantities. For example,  $L = \lambda/i = N \sum_i \frac{1}{\mathcal{R}_i}$  for a singly-excited system.

And now the term, "reluctance motor", finally means something. The idea is to get a force based on the tendency for the system to adjust so as to minimize energy. But if energy is  $w_m = \frac{1}{2} \sum_i \mu_i H_i^2 A_i \ell_i$ , and if  $H_i = B_i/\mu_i = \lambda_i/(\mu_i A_i)$ , then  $w_m = \frac{1}{2} \sum_i \lambda_i^2 \mathcal{R}_i = \frac{1}{2} \sum_i (NI)^2/\mathcal{R}_i$ , the system where  $\lambda$  is prescribed (by a voltage source) has minimum energy for minimum reluctance.

Also, more complex systems are now readily handled by applying the normal parallel and series combination rules for reluctances that are used for resistances. So we now have a crank-turn procedure for arbitrary magneto-static configurations provided we can always neglect fringing fields.

## A.2.2 Part b: A (Slightly) More Mathematically Formal Discussion on Calculating Forces

A differential amount of mechanical work,  $dw$ , is defined through  $\vec{f} \cdot d\vec{x}$ . By convention, we say that work is done *by* the force field *on* an object when  $dw > 0$  (i.e. when the differential movement is in the direction of the force), and that *external* work is done *on* the field when  $dw < 0$  (i.e. when the motion opposes the force's direction). In the latter case, the energy is stored as potential in the field. The potential energy is defined for the latter situation from an arbitrary reference position as  $U = - \int_{\vec{r}_0}^{\vec{r}} \vec{f} \cdot d\vec{r}$ , where the negative sign appears such that the potential energy is a positive quantity when the path direction opposes that of the force. Going in the other direction leads to a specification of the force if the potential is known,  $\vec{f} = -\nabla U$ . This *always* works when the system is curl-free (i.e.  $\nabla \times \rightarrow 0$ ), which, in integral form, says that any line integral which starts and ends in the same place is zero. This makes the potential a *scalar, conservative* field. Whenever you do energy calculations and reach a conceptual standstill, you can always return to these definitions, especially in the application of a gravitational potential (to which you have grown accustomed since birth).

A confusing point to the energy calculations that we do in 6.013 is that we intuitively know that the force acts to bring the pole faces together and minimize reluctance, but our calculations seem to show that this tends to

make the fields, and therefore the stored energy, stronger (i.e. we seem to be maximizing energy instead of minimizing it). The confusion is exactly analogous to the situation of  $P = V^2/R = I^2R$  (remembering that total magnetic energy plays a similar role in magnetostatic calculations that power does in quasielectrostatic calculations). When voltage is prescribed,  $P$  is minimized for maximum  $R$ ; when current is prescribed, it is minimized for minimum  $R$ . The same is true of magnetostatic circuits, substituting  $w_m$  for  $P$ ,  $NI$  for  $V$ ,  $\mathcal{R}$  for  $R$ , and  $\lambda$  for  $I$ . Confusingly, where we normally think of  $V$  as specified in a quasielectrostatic circuit, it is  $\lambda$  that is typically specified in a magnetostatic circuit<sup>1</sup>. And  $\mathcal{R}$  is minimized when the gap length is minimized, so the pole faces tend to pull together.

As usual, significant words and ideas can be hidden in a few deceptively simple formulations.

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<sup>1</sup>Ironically, this is so *because* the voltage is specified in the driving electrical circuit.

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