Problem Set 3 Solutions

Problem 3.1

(a) Using the force/energy equation, we know that the force attracting the two plates will be equal to the partial derivative of the stored energy in the system with respect to the changing variable (in this case, d).

$f_\perp = \frac{\partial w_e}{\partial d}$

We know that the energy stored in a capacitor with a fixed charge Q on the plates is $\omega_e = \frac{Q^2 d}{2\varepsilon A}$, so the force attracting the two plates is

$f_\perp = \frac{\partial w_e}{\partial d} = \frac{\partial}{\partial d}\frac{Q^2 d}{2\varepsilon A} = \frac{Q^2}{2\varepsilon A}$

Noting that $Q = \frac{\varepsilon AV}{d}$ we can cast the result in terms of the applied voltage

$$f_\perp = \frac{V^2 \varepsilon_\circ bL}{2d^2} \quad (1)$$

It's also possible to solve this problem using the pressure due to the electric field on the conductor, $\bar{f} = -P_e A$. We know $P_e = \frac{-\varepsilon_\circ E_\circ^2}{2}$, and we can relate E and A to the voltage and dimensions of the overlap.

$f = -P_e A = \frac{\varepsilon_\circ V^2}{2d^2} bL = \frac{V^2 \varepsilon_\circ bL}{2d^2}$

Which is the same as the result using the force/energy relation.

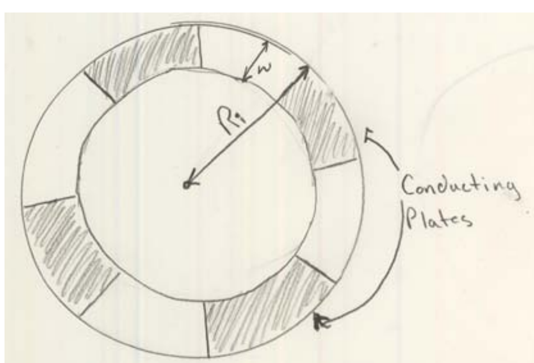
(b) Again we can use the force/energy equation but in this case the derivative is taken with respect to b, and has a negative sign since pulling the plates apart horizontally decreases b.

$f_\parallel = -\frac{\partial w_e}{\partial b} = -\frac{\partial}{\partial b}\frac{Q^2 d}{2\varepsilon L b} = \frac{Q^2 d}{2\varepsilon L b^2}$

Again we can cast this in terms of the voltage on the capacitor using $Q = \frac{\varepsilon AV}{d}$

$$f_\parallel = \frac{V^2 \varepsilon_\circ L}{2d} \quad (2)$$

(c) Consider a ring with outer radius R, and width w, having N segments ($N/2$ segments are conducting plates, see diagram below).



By these conditions, each segment has angular width

$\Theta_{width} = \frac{2\pi}{N}$

If we place this ring above an identical ring, than the area of overlap of conducting plates will go from zero to $(2wR - w^2)\pi$ as Θ goes from zero to Θ_{width}. So we can express the overlap area as a function of Θ by

$A = (2wR - w^2)\frac{N}{4}\Theta = A_\circ\Theta$,

where $A_\circ$ is a constant (with respect to Θ). Using this expression, we can calculate capacitance as a function of Θ,

$C = \frac{\varepsilon A}{d} = \frac{\varepsilon A_\circ}{s}\Theta = C_\circ\Theta$

where d is the separation between the two rings.

The stored energy in the overlap is

$w_e = \frac{Q^2}{2C} = \frac{Q^2}{2C_\circ\Theta}$

giving a torque

$T = -\frac{\partial w_e}{\partial\Theta} = \frac{-Q^2}{2C_\circ}\frac{\partial}{\partial\Theta}\Theta^{-1} = \frac{Q^2 2d}{\varepsilon N(2wR-w^2)\Theta^2}$

If we take the case of one ring, with $w = R$ and $N = 4$, then the above expression becomes equation 6.2.14 from the notes.

From the notes, we know that the power from the %50 duty cycle is $P = T\omega/2$. We know that the angular velocity will be limited by the maximum speed of the outer edge, $w_{max} = v_{max}/R$. So the mechanical power obtained from one ring is

$$Power = \frac{(2wR - w^2)N\varepsilon_\circ E_{bd}^2 d v_{max}}{16R} \tag{3}$$

To get the highest power, we want to put as many segments as possible into the total allowed volume. This will happen when we make w and d as small as possible and R as large as possible ($d = 0.5$ $[mm]$, $w = 4d = 2$ $[mm]$).

$vol = 1\ [L] = 0.001\ [m^3] = d\pi(R^2 - (R - w)^2) = d\pi(2Rw - w^2) = \pi(R8d^2 - 16d^3)$

$R = \frac{0.001}{8\pi d^2} + 4d = 159.16\ [m]$

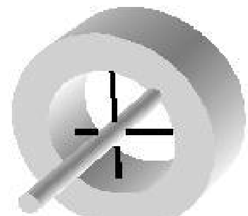
If we limit the length of any one segment to 4d to reduce the fringing fields, we can fit $N = \frac{2\pi R}{4d} = 500015.8$ (but we need N to be an even whole number, so choose $N = 500014$) segments on the ring. So the total mechanical power becomes:

$Power = \frac{(2wR-w^2)N\varepsilon_\circ E_{bd}^2 d v_{max}}{16R} = \frac{(2\times0.002\times159.16-16\times(0.002)^2)\times500014\times(8.85\times10^{-12})\times(0.0005)\times100}{16\times159.16}E_{bd}^2$

$Power = 5.53\times10^{-11}\ E_{bd}^2$

For $E_{bd} = 10^6\ [V/m]$ this becomes $Power = 55.3\ [W]$

(d) To get the maximum power we want to have a high number of segments, but to reduce fringing fields there is a limit to how small we can make any individual segment. Since we were not told how to make the structure, we put all the volume into a single ring with a 159 m radius. This neglects any material used to drive and support the structure. See diagram below.



(e) $Power = 55.3$ $[W]$, and the ring has a radius of 159.16 $[m]$.

Problem 3.2

(a) From the Lorentz force law, we know that electrons moving with velocity $\bar{v}$ will experience a force $\bar{f} = -e(\bar{E} + \bar{v} \times \mu_\circ \bar{H})$. In the case of a conductor moving through a magnetic field, this force will cause electrons to move within the conductor until the electric forces due to the distribution of electrons balance the magnetic forces due to the conductors motion.

Using the notation in the following diagram, we can pretend that Io is moving through the magnetic field $\bar{H} = -\hat{y}\ \frac{10^{-4}}{\mu_\circ}$ $[A/m]$ with velocity $\bar{v} = \hat{z}\ 7200[m/s]$.

If the electric and magnetic forces are balanced, than f is zero, so we can solve for the electric field in terms of the magnetic field.

$\bar{f} = -e(\bar{E} + \bar{v} \times \mu_\circ \bar{H}) = 0$

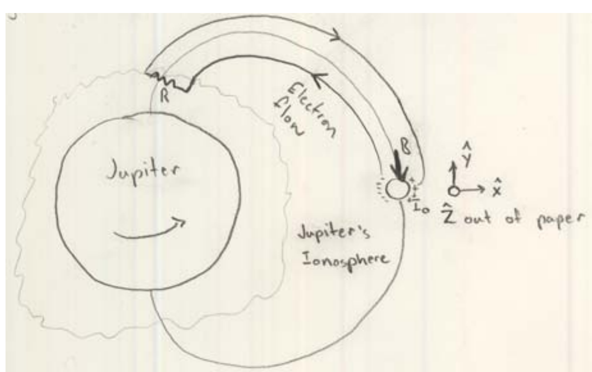
$\bar{E} = -\bar{v} \times \mu_\circ \bar{H} = -\hat{z}\ v \times -\hat{y}\ \mu_\circ H$

$\bar{E} = -\hat{x}\mu_\circ v H$

Noting that the field is constant within the conductor, we know that the potential difference V across the conductor will be the product of the field and the length of the conductor, $V = Ed$ (in this case, d is the width of Io).

$V = d\mu_\circ v H = (3340 \times 10^3)(7200)(10^{-4}) = 2.4 \times 10^6$ $[V]$.

Knowing the direction of the induced electric field, $-\hat{x}$, we can determine that the side of Io closest to Jupiter will accumulate a negative charge (if no other current path exists).



(b) Electrons will flow from the side of Io closest Jupiter, along the magnetic field lines, through Jupiter's ionosphere, and back to the Io. Through Io this results in a flow of electrons along $-\hat{x}$ (see diagram), with a current

$\bar{I} = \hat{x}\frac{2.4 \times 10^6 [V]}{1\Omega}$

The force exerted on Io by the magnetic field, due to the current, will be

$\bar{f} = \bar{I} \times \bar{H}\mu_\circ d = \hat{x}\frac{2.4\times10^6[V]}{1\Omega} \times -\hat{y}(10^{-4})(3340 \times 10^3)$

$\bar{f} = -\hat{z}\ 801.6 \times 10^6[N]$

This force speeds up Io's movement around Jupiter.

(c) $P = \frac{V^2}{R} = \frac{(2.4\times10^6)^2}{1} = 5.76 \times 10^{12}\ [Watts]$

Problem 3.3

Ignoring Relativistic effects

The maximum range of the device is twice the radius associated with cyclotron motion of the 10^6 eV kinetic energy electron in a 10^{-5} Tesla magnetic field.

$ElectronEnergy = 10^6 * e = \frac{1}{2}mv^2\ [J]$

$v = \sqrt{\frac{2*10^6 e}{m_e}} = \omega_e * R$, so $R = \frac{1}{\omega_e}\sqrt{\frac{2*10^6 e}{m_e}}$

The cyclotron frequency is defined by

$\omega_e = \frac{e\mu_\circ H}{m_e}$

Giving a maximum range of

$Range = 2R = 2\frac{m_e}{e\mu_\circ H}\sqrt{\frac{2\times10^6 e}{m_e}} = \frac{2}{\mu_\circ H}\sqrt{\frac{2\times10^6 m_e}{e}}$

$$Range = \frac{2}{1.26\times10^{-6}\times10^{-5}}\sqrt{\frac{2\times10^6\times9.1\times10^{-31}}{1.6\times10^{-19}}} = 5.38\times10^8\ [m] \qquad (4)$$

Dealing with Relativistic effects

If we have to deal with relativistic effects, the kinetic energy of the electron can be expressed as

$E_k = m_e c^2\left(\frac{1}{\sqrt{1-(v/c)^2}} - 1\right)$, giving an expression for the velocity of the electron

$v = c\sqrt{1 - (\frac{E_k}{m_e c^2} + 1)^{-2}} = 2.8 \times 10^8\ [m/s]$

The cyclotron radius associated with this particle will be

$R = \frac{1}{\sqrt{1-(v/c)^2}}\frac{m_e v}{e\mu_\circ H}$

which we can evaluate to get a maximum range of $7 \times 10^8\ [m]$.

Problem 3.4

(a) To calculate the inductance of the device, we first need to find the magnitude of the H field in the device. Using Ampere's law we can relate the magnitude of the H field (inside the iron and in the gap) to the current through the N turns of the wire on the iron circle. This is equation 3.2.19 in the notes. Applying the relation $\mu H_\mu = \mu_\circ H_g$ (from the fact that perpendicular B field is continuous from the iron into the gap) to equation 3.2.19 we can find the magnitude of the H field in the gap.

$H_g[D\frac{\mu_\circ}{\mu} + d] = Ni$

$H_g = Ni[D\frac{\mu_\circ}{\mu} + d]^{-1}$

If we assume that the gap stores nearly all the energy in the device, we can reduce this expression to

$H_g = \frac{Ni}{d}$.

Using this H field in equation 3.2.16 gives the inductance of the device as

$$L = \frac{\mu_\circ N \int\int_A \bar{H} \bullet d\bar{a}}{i} = \frac{\mu_\circ N^2 A}{d} \tag{5}$$

(b) To find the force pulling the gap closed apply the force energy equation to the energy stored in the device. In the capacitor case, the constant was total charge on one of the plates. In the case for the inductor we'll look at constant $\Lambda = Li$ (short circuit the coil while still allowing current to flow). The energy stored in an inductor is $w_m = \frac{\Lambda^2}{2L}[J]$ by equation 3.2.23 and 3.2.16. So, the force pulling the gap closed is

$f = \frac{\partial w_m}{\partial d} = \frac{\partial}{\partial d}\frac{\Lambda^2}{2L} = \frac{\Lambda^2}{2}\frac{\partial}{\partial d}\frac{d}{\mu_\circ N^2 A} = \frac{\Lambda^2}{2\mu_\circ N^2 A}$

Which reduces to:

$$f = \frac{\mu_\circ N^2 A i^2}{2d^2} \tag{6}$$

It is also possible to solve this problem using the pressure on the surface of the rod at the gap due to the magnetic field.

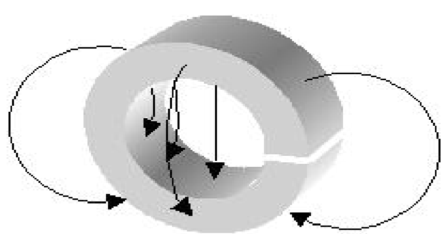
$f = P_m A = \frac{\mu_\circ H^2}{2} A = \frac{\mu_\circ A (Ni)^2}{2d^2} A = \frac{\mu_\circ N^2 A i^2}{2d^2}$

Which is the same result as above.

(c) When $d = 0$ there is no gap, so the H field inside the iron cylinder becomes $H = H_\mu = \frac{Ni}{D}$ giving an inductance of

$$L = \frac{\mu N^2 A}{D} \tag{7}$$

(d) The $\bar{B}$ field emerges from the high permeability torus nearly at right angles, and therefore representative $\bar{B}$ field lines might include those illustrated here (plus others, not drawn).



(e) From equation 3.2.13 we know that the voltage across the coil is the derivative of the flux linkage, Λ. The flux linkage is

$\Lambda = Li = \frac{\mu_\circ N^2 A}{d} i$

We can express d as a function of time using

$d(t) = d(1 + \frac{t}{\tau})$

giving $\frac{\partial d(t)}{\partial t} = \frac{d}{\tau}$

so the volatage across the coil will be

$$V = \frac{\partial \Lambda}{\partial t} = \frac{\partial}{\partial t} i \frac{\mu_\circ N^2 A}{d(t)} = \frac{-i\mu_\circ N^2 A}{d\tau(1+t/\tau)^2} \tag{8}$$

for $0 < t < \tau$.

Note that this voltage jumps from 0 at $t = 0^-$ to some value at $t = 0^+$.

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