**Massachusetts Institute of Technology**
**Department of Electrical Engineering and Computer Science**
**6.013 FORMULA SHEET**

## 1. DIFFERENTIAL OPERATORS IN CYLINDRICAL AND SPHERICAL COORDINATES

**If r, $\phi$, and z are circular cylindrical coordinates and $\hat{\imath}_r$, $\hat{\imath}_\phi$, and $\hat{\imath}_z$ are unit vectors in the directions of increasing values of the corresponding coordinates,**

$$\nabla U = grad\, U = \hat{\imath}_r \frac{\partial U}{\partial r} + \hat{\imath}_\phi \frac{1}{r}\frac{\partial U}{\partial \phi} + \hat{\imath}_z \frac{\partial U}{\partial z}$$

$$\nabla \cdot \vec{A} = div\vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \vec{A} = curl\vec{A} = \hat{\imath}_r \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\imath}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\imath}_z \left(\frac{1}{r}\frac{\partial (rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right)$$

$$\nabla^2 U = div\, grad\, U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

**If r, $\theta$, and $\phi$ are spherical coordinates and $\hat{\imath}_r$, $\hat{\imath}_\theta$, and $\hat{\imath}_\phi$ are unit vectors in the directions of increasing values of the corresponding coordinates,**

$$\nabla U = grad\, U = \hat{\imath}_r \frac{\partial U}{\partial r} + \hat{\imath}_\theta \frac{1}{r}\frac{\partial U}{\partial \theta} + \hat{\imath}_\phi \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}$$

$$\nabla \cdot \vec{A} = div\vec{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \vec{A} = curl\vec{A} = \hat{\imath}_r \left(\frac{1}{r\sin\theta}\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi}\right) + \hat{\imath}_\theta \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}\right) + \hat{\imath}_\phi \left(\frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$

$$\nabla^2 U = div\, grad\, U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}$$

## 2. SOLUTIONS OF LAPLACE'S EQUATIONS

### A. Rectangular coordinates, two dimensions (independent of z):

$$\Phi = e^{kx}(A_1 \sin ky + A_2 \cos ky) + e^{-kx}(B_1 \sin ky + B_2 \cos ky)$$

**(or replace $e^{kx}$ and $e^{-kx}$ by $\sinh kx$ and $\cosh kx$).**

$$\Phi = Axy + Bx + Cy + D; (k = 0)$$

### B. Cylindrical coordinates, two dimensions (independent of z):

$$\Phi = r^n (A_1 \sin n\phi + A_2 \cos n\phi) + r^{-n}(B_1 \sin n\phi + B_2 \cos n\phi)$$

$$\Phi = \ln\frac{R}{r}(A_1\phi + A_2) + B_1\phi + B_2; (n = 0)$$

### C. Spherical coordinates, two dimensions (independent of $\phi$):

$$\Phi = Ar\cos\theta + \frac{B}{r^2}\cos\theta + \frac{C}{r} + D$$