6.013 Formula Sheet # 1

Cartesian Coordinates (x, y, z)

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{i}_y + \frac{\partial f}{\partial z}\mathbf{i}_z$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{i}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{i}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{i}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Cylindrical Coordinates (r, ϕ, z)

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi + \frac{\partial f}{\partial z}\mathbf{i}_z$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{i}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{i}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{i}_z\frac{1}{r}\left[\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Coordinates (r, θ, ϕ)

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \mathbf{i}_r\frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right] + \mathbf{i}_\theta\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial (rA_\phi)}{\partial r}\right] + \mathbf{i}_\phi\frac{1}{r}\left[\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Cartesian		*Cylindrical*		*Spherical*
x	=	$r\cos\phi$	=	$r\sin\theta\cos\phi$
y	=	$r\sin\phi$	=	$r\sin\theta\sin\phi$
z	=	z	=	$r\cos\theta$
$\mathbf{i}_x$	=	$\cos\phi\mathbf{i}_r - \sin\phi\mathbf{i}_\phi$	=	$\sin\theta\cos\phi\mathbf{i}_r + \cos\theta\cos\phi\mathbf{i}_\theta - \sin\phi\mathbf{i}_\phi$
$\mathbf{i}_y$	=	$\sin\phi\mathbf{i}_r + \cos\phi\mathbf{i}_\phi$	=	$\sin\theta\sin\phi\mathbf{i}_r + \cos\theta\sin\phi\mathbf{i}_\theta + \cos\phi\mathbf{i}_\phi$
$\mathbf{i}_z$	=	$\mathbf{i}_z$	=	$\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_\theta$

Cylindrical		*Cartesian*		*Spherical*
r	=	$\sqrt{x^2+y^2}$	=	$r\sin\theta$
ϕ	=	$\tan^{-1}\frac{y}{x}$	=	ϕ
z	=	z	=	$r\cos\theta$
$\mathbf{i}_r$	=	$\cos\phi\mathbf{i}_x + \sin\phi\mathbf{i}_y$	=	$\sin\theta\mathbf{i}_r + \cos\theta\mathbf{i}_\theta$
$\mathbf{i}_\phi$	=	$-\sin\phi\mathbf{i}_x + \cos\phi\mathbf{i}_y$	=	$\mathbf{i}_\phi$
$\mathbf{i}_z$	=	$\mathbf{i}_z$	=	$\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_\theta$

Spherical		*Cartesian*		*Cylindrical*
r	=	$\sqrt{x^2+y^2+z^2}$	=	$\sqrt{r^2+z^2}$
θ	=	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	=	$\cos^{-1}\frac{z}{\sqrt{r^2+z^2}}$
ϕ	=	$\cot^{-1}\frac{x}{y}$	=	ϕ
$\mathbf{i}_r$	=	$\sin\theta\cos\phi\mathbf{i}_x + \sin\theta\sin\phi\mathbf{i}_y + \cos\theta\mathbf{i}_z$	=	$\sin\theta\mathbf{i}_r + \cos\theta\mathbf{i}_z$
$\mathbf{i}_\theta$	=	$\cos\theta\cos\phi\mathbf{i}_x + \cos\theta\sin\phi\mathbf{i}_y - \sin\theta\mathbf{i}_z$	=	$\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_z$
$\mathbf{i}_\phi$	=	$-\sin\phi\mathbf{i}_x + \cos\phi\mathbf{i}_y$	=	$\mathbf{i}_\phi$

Geometric relations between coordinates and unit vectors for Cartesian, cylindrical, and spherical coordinate systems.

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Vector Identities

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}=\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=(\mathbf{C}\times\mathbf{A})\cdot\mathbf{B}$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$
$$\nabla\cdot(\nabla\times\mathbf{A})=0$$
$$\nabla\times(\nabla f)=0$$
$$\nabla(fg)=f\nabla g+g\nabla f$$
$$\nabla(\mathbf{A}\cdot\mathbf{B})=(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})$$
$$\nabla\cdot(f\mathbf{A})=f\nabla\cdot\mathbf{A}+(\mathbf{A}\cdot\nabla)f$$
$$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$$
$$\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$$
$$\nabla\times(f\mathbf{A})=\nabla f\times\mathbf{A}+f\nabla\times\mathbf{A}$$
$$(\nabla\times\mathbf{A})\times\mathbf{A}=(\mathbf{A}\cdot\nabla)\mathbf{A}-\frac{1}{2}\nabla(\mathbf{A}\cdot\mathbf{A})$$
$$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$$

Integral Theorems

Line Integral of a Gradient

$$\int_a^b \nabla f\cdot\mathbf{dl}=f(b)-f(a)$$

Divergence Theorem:

$$\int_V \nabla\cdot\mathbf{A}dV=\oint_S \mathbf{A}\cdot\mathbf{dS}$$

Corollaries

$$\int_V \nabla f dV=\oint_S f\mathbf{dS}$$
$$\int_V \nabla\times\mathbf{A}dV=-\oint_S \mathbf{A}\times\mathbf{dS}$$

Stokes' Theorem:

$$\oint_L \mathbf{A}\cdot\mathbf{dl}=\int_S(\nabla\times\mathbf{A})\cdot\mathbf{dS}$$

Corollary

$$\oint_L f\mathbf{dl}=-\int_S \nabla f\times\mathbf{dS}$$

Maxwell's Equations

Integral	*Differential*	*Boundary Conditions*
Faraday's Law		
$\oint_L \mathbf{E}'\cdot\mathbf{dl}=-\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{dS}$	$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$	$\mathbf{n}\times(\mathbf{E}_2'-\mathbf{E}_1')=0$
Ampere's Law with Maxwell's Displacement Current Correction		
$\oint_L \mathbf{H}\cdot\mathbf{dl}=\int_S \mathbf{J}_f\cdot\mathbf{dS}+\frac{d}{dt}\int_S \mathbf{D}\cdot\mathbf{dS}$	$\nabla\times\mathbf{H}=\mathbf{J}_f+\frac{\partial\mathbf{D}}{\partial d}$	$\mathbf{n}\times(\mathbf{H}_2-\mathbf{H_1})=\mathbf{K}_f$
Gauss's Law		
$\oint_S \mathbf{D}\cdot\mathbf{dS}=\int_V \rho_f dV$	$\nabla\cdot\mathbf{D}=\rho_f$	$\mathbf{n}\cdot(\mathbf{D}_2-\mathbf{D}_1)=\sigma_f$
$\oint_S \mathbf{B}\cdot\mathbf{dS}=0$	$\nabla\cdot\mathbf{B}=0$	$\mathbf{n}\cdot(\mathbf{B}_2-\mathbf{B}_1)=0$
Conservation of Charge		
$\oint_S \mathbf{J}_f\cdot\mathbf{dS}+\frac{d}{dt}\int_V \rho_f dV=0$	$\nabla\cdot\mathbf{J}_f+\frac{\partial\rho_f}{\partial t}=0$	$\mathbf{n}\cdot(\mathbf{J}_2-\mathbf{J}_1)+\frac{\partial\sigma_f}{\partial t}=0$

Usual Linear Constitutive Laws

$\mathbf{D}=\epsilon\mathbf{E}$

$\mathbf{B}=\mu\mathbf{H}$

$\mathbf{J_f}=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B})=\sigma\mathbf{E}'$ [Ohm's law for moving media with velocity $\mathbf{v}$]

Physical Constants

Constant	Symbol	Value	units
Speed of light in vacuum	c	$2.9979\times10^8\approx3\times10^8$	m/sec
Elementary electron charge	e	1.602×10^{-19}	coul
Electron rest mass	m_e	9.11×10^{-31}	kg
Electron charge to mass ratio	$\frac{e}{m_e}$	1.76×10^{11}	coul/kg
Proton rest mass	m_p	1.67×10^{-27}	kg
Boltzmann constant	k	1.38×10^{-23}	joule/°K
Gravitation constant	G	6.67×10^{-11}	nt-m^2/(kg)^2
Acceleration of gravity	g	9.807	m/(sec)^2
Permittivity of free space	ϵ_0	$8.854\times10^{-12}\approx\frac{10^{-9}}{36\pi}$	farad/m
Permeability of free space	μ_0	$4\pi\times10^{-7}$	henry/m
Planck's constant	h	6.6256×10^{-34}	joule-sec
Impedance of free space	$\eta_0=\sqrt{\frac{\mu_0}{\epsilon_0}}$	$376.73\approx120\pi$	ohms
Avogadro's number	N	6.023×10^{23}	atoms/mole