| **6.013 - Electromagnetics and Applications** | Fall 2005 |
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# Lecture 8 - Transverse Electromagnetic Waves

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## I. Maxwell's Equations for Linear Media

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t}$$
$$\nabla \times \bar{H} = \bar{J} + \epsilon \frac{\partial \bar{E}}{\partial t}$$
$$\nabla \cdot \bar{E} = \rho_f / \epsilon$$
$$\nabla \cdot \bar{H} = 0$$

## II. Poynting's Theorem

### A. Power Flow, Electromagnetic Energy

$$\begin{aligned}
\nabla \cdot \left(\bar{E} \times \bar{H}\right) &= \bar{H} \cdot \left(\nabla \times \bar{E}\right) - \bar{E} \cdot \left(\nabla \times \bar{H}\right) \\
&= -\mu \bar{H} \cdot \frac{\partial \bar{H}}{\partial t} - \bar{E} \cdot \left[\bar{J} + \epsilon \frac{\partial \bar{E}}{\partial t}\right] \\
&= -\frac{\mu}{2} \frac{\partial}{\partial t}\left(|\bar{H}|^2\right) - \frac{\epsilon}{2} \frac{\partial}{\partial t}\left(|\bar{E}|^2\right) - \bar{E} \cdot \bar{J}
\end{aligned}$$

$$\nabla \cdot \left(\bar{E} \times \bar{H}\right) + \frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon |\bar{E}|^2 + \frac{1}{2}\mu |\bar{H}|^2\right] = -\bar{E} \cdot \bar{J}$$

$$\int_V \nabla \cdot \left(\bar{E} \times \bar{H}\right) dV = \oint_S \bar{E} \times \bar{H} \cdot d\bar{S}$$

$$\oint_S \bar{E} \times \bar{H} \cdot d\bar{a} + \frac{d}{dt} \int_V \left(\frac{1}{2}\epsilon |\bar{E}|^2 + \frac{1}{2}\mu |\bar{H}|^2\right) dV = -\int_V \bar{E} \cdot \bar{J}\, dV$$

$\bar{S} = \bar{E} \times \bar{H}$ — Poynting Vector (watts / m$^2$)

$W = \int_V \left(\frac{1}{2}\epsilon |\bar{E}|^2 + \frac{1}{2}\mu |\bar{H}|^2\right) dV$ — Electromagnetic Stored Energy

$P_d = \int_V \bar{E} \cdot \bar{J}\, dV$ — Power dissipated if $\bar{J} \cdot \bar{E} > 0$

*e.g.*, $\bar{J} = \sigma \bar{E} \Rightarrow \bar{J} \cdot \bar{E} = \sigma |\bar{E}|^2$

Power source if $\bar{J} \cdot \bar{E} < 0$

$P_{\text{out}} = \oint_S \bar{E} \times \bar{H} \cdot d\bar{a} = \oint_S \bar{S} \cdot d\bar{a}$

$P_{\text{out}} + \frac{dW}{dt} = -P_d$

$w_e = \frac{1}{2}\epsilon |\bar{E}|^2$ — Electric energy density in Joules/m$^3$

$w_m = \frac{1}{2}\mu |\bar{H}|^2$ — Magnetic energy density in Joules/m$^3$

B. Power in Electric Circuits

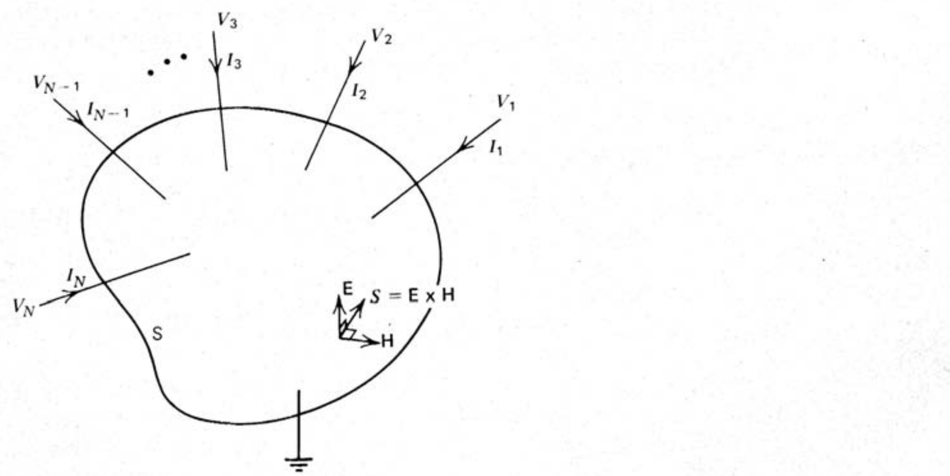


Figure 7-2 The circuit power into an $N$ terminal network $\sum_{k=1}^{N} V_k I_k$ equals the electromagnetic power flow into the surface surrounding the network, $-\oint_S \mathbf{E}\times\mathbf{H}\cdot\mathbf{dS}$.



Outside circuit elements

$$\oint_C \bar{E}\cdot d\bar{l} \approx 0, \qquad \nabla\times\bar{E} = 0 \Rightarrow \bar{E} = -\nabla\Phi \quad \text{(Kirchoff's Voltage Law } \textstyle\sum_k v_k = 0$$

$$\nabla\times\bar{H} = \bar{J} \Rightarrow \nabla\cdot\bar{J} = 0, \qquad \oint_S \bar{J}\cdot d\bar{S} = 0 \quad \text{(Kirchoff's current law } \textstyle\sum_k i_k = 0$$

$$\begin{aligned} P_{\text{in}} &= -\oint_S \bar{E}\times\bar{H}\cdot d\bar{S} \\ &= -\int_V \nabla\cdot\left(\bar{E}\times\bar{H}\right)\, dV \end{aligned}$$

$$\nabla\cdot\left(\bar{E}\times\bar{H}\right) = \bar{H}\cdot\cancelto{0}{\nabla\times\bar{E}} - \bar{E}\cdot\left(\nabla\times\bar{H}\right) = -\bar{E}\cdot\bar{J} = \nabla\Phi\cdot\bar{J}$$

$$\nabla\cdot\left(\bar{J}\Phi\right) = \Phi\cancelto{0}{\nabla\cdot\bar{J}} + \bar{J}\cdot(\nabla\Phi)$$

$$\nabla\cdot\left(\bar{E}\times\bar{H}\right) = \bar{J}\cdot(\nabla\Phi) = \nabla\cdot\left(\Phi\bar{J}\right)$$

$$P_{\text{in}} = -\int_V \nabla\cdot\left(\bar{E}\times\bar{H}\right) dV = -\int_V \nabla\cdot\left(\bar{J}\Phi\right)\, dV = -\oint_S \bar{J}\Phi\cdot d\bar{S}$$

On $S, \Phi =$ voltages on each wire, $\bar{J}$ is non-zero only on wires.

$$P_{\text{in}} = -\oint_S \bar{J}\Phi\cdot d\bar{S} = -\sum_{k=1}^{N} v_k \underbrace{\oint_S \bar{J}\cdot d\bar{S}}_{-i_k} = \sum_{k=1}^{N} v_k i_k$$

C. Complex Poynting's Theorem (Sinusoidal Steady State, $e^{j\omega t}$)

$$\bar{E}(\bar{r},t) = \text{Re}\left[\hat{\bar{E}}(\bar{r})e^{j\omega t}\right] = \frac{1}{2}\left[\hat{\bar{E}}(\bar{r})e^{j\omega t} + \hat{\bar{E}}^*(\bar{r})e^{-j\omega t}\right]$$

$$\bar{H}(\bar{r},t) = \text{Re}\left[\hat{\bar{H}}(\bar{r})e^{j\omega t}\right] = \frac{1}{2}\underbrace{\left[\hat{\bar{H}}(\bar{r})e^{j\omega t} + \hat{\bar{H}}^*(\bar{r})e^{-j\omega t}\right]}_{\substack{\text{The real part of a complex number is} \\ \text{one-half of the sum of the number and its} \\ \text{complex conjugate}}}$$

Maxwell's Equations in Sinusoidal Steady State

$$
\begin{aligned}
\nabla \times \hat{\bar{E}}(\bar{r}) &= -j\omega\mu\hat{\bar{H}}(\bar{r}) \\
\nabla \times \hat{\bar{H}}(\bar{r}) &= \bar{J}(\bar{r}) + j\omega\epsilon\hat{\bar{E}}(\bar{r}) \\
\nabla \cdot \hat{\bar{E}}(\bar{r}) &= \hat{\rho}_f(\bar{r})/\epsilon \\
\nabla \cdot \hat{\bar{H}}(\bar{r}) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\bar{S}(\bar{r},t) &= \bar{E}(\bar{r},t) \times \bar{H}(\bar{r},t) \\
&= \frac{1}{4}\left[\hat{\bar{E}}(\bar{r})e^{j\omega t} + \hat{\bar{E}}^*(\bar{r})e^{-j\omega t}\right] \times \left[\hat{\bar{H}}(\bar{r})e^{j\omega t} + \hat{\bar{H}}^*(\bar{r})e^{-j\omega t}\right] \\
&= \frac{1}{4}\left[\hat{\bar{E}}(\bar{r}) \times \hat{\bar{H}}(\bar{r})e^{2j\omega t} + \hat{\bar{E}}^*(\bar{r}) \times \hat{\bar{H}}(\bar{r}) + \hat{\bar{E}}(\bar{r}) \times \hat{\bar{H}}^*(\bar{r}) + \hat{\bar{E}}^*(\bar{r}) \times \hat{\bar{H}}^*(\bar{r})e^{-2j\omega t}\right] \\
\langle \bar{S} \rangle &= \frac{1}{4}\left[\hat{\bar{E}}^*(\bar{r}) \times \hat{\bar{H}}(\bar{r}) + \hat{\bar{E}}(\bar{r}) \times \hat{\bar{H}}^*(\bar{r})\right] \\
&= \frac{1}{2}\,\mathrm{Re}\left[\hat{\bar{E}}(\bar{r}) \times \hat{\bar{H}}^*(\bar{r})\right] = \frac{1}{2}\,\mathrm{Re}\left[\hat{\bar{E}}^*(\bar{r}) \times \hat{\bar{H}}(\bar{r})\right]
\end{aligned}
$$

(A complex number plus its complex conjugate is twice the real part of that number.)

$$
\hat{\bar{S}} = \frac{1}{2}\hat{\bar{E}}(\bar{r}) \times \hat{\bar{H}}(\bar{r})^*
$$

$$
\begin{aligned}
\nabla \cdot \hat{\bar{S}} = \nabla \cdot \left[\frac{1}{2}\hat{\bar{E}}(\bar{r}) \times \hat{\bar{H}}^*(\bar{r})\right] &= \frac{1}{2}\left[\hat{\bar{H}}^*(\bar{r}) \cdot \nabla \times \hat{\bar{E}}(\bar{r}) - \hat{\bar{E}}(\bar{r}) \cdot \nabla \times \hat{\bar{H}}^*(\bar{r})\right] \\
&= \frac{1}{2}\left[\hat{\bar{H}}^*(\bar{r})\left(-j\omega\mu\hat{\bar{H}}(\bar{r})\right) - \hat{\bar{E}}(\bar{r}) \cdot \left(\hat{\bar{J}}^*(\bar{r}) - j\omega\epsilon\hat{\bar{E}}^*(\bar{r})\right)\right] \\
&= \frac{1}{2}\left[-j\omega\mu|\hat{\bar{H}}(\bar{r})|^2 + j\omega\epsilon|\hat{\bar{E}}(\bar{r})|^2\right] - \frac{1}{2}\hat{\bar{E}}(\bar{r}) \cdot \hat{\bar{J}}^*(\bar{r})
\end{aligned}
$$

$$
\begin{aligned}
\langle w_m \rangle = \frac{1}{4}\mu|\hat{\bar{H}}(\bar{r})|^2, \langle w_e \rangle &= \frac{1}{4}\epsilon|\hat{\bar{E}}(\bar{r})|^2 \\
\hat{P}_d &= \frac{1}{2}\hat{\bar{E}}(\bar{r}) \cdot \hat{\bar{J}}^*(\bar{r}) \\
\nabla \cdot \hat{\bar{S}} + 2j\omega\left[\langle w_m \rangle - \langle w_e \rangle\right] &= -\hat{P}_d
\end{aligned}
$$

III. Transverse Electromagnetic Waves ($\rho_f = 0, \bar{J} = 0$)

A. Wave equation

$$
\begin{aligned}
\nabla \times \bar{E} &= -\mu\frac{\partial \bar{H}}{\partial t} \\
\nabla \times \bar{H} &= \epsilon\frac{\partial \bar{E}}{\partial t} \\
\nabla \cdot \bar{E} &= 0 \\
\nabla \cdot \bar{H} &= 0
\end{aligned}
$$

$$\nabla \times (\nabla \times \bar{E}) = -\mu \frac{\partial}{\partial t} (\nabla \times \bar{H}) = -\mu \frac{\partial}{\partial t} \left( \epsilon \frac{\partial \bar{E}}{\partial t} \right)$$

$$\nabla \times (\nabla \times \bar{E}) = \nabla \left( \cancelto{0}{\nabla \cdot \bar{E}} \right) - \nabla^2 \bar{E} = -\epsilon\mu \frac{\partial^2 \bar{E}}{\partial t^2}$$

Wave equation

$$\nabla^2 \bar{E} = \frac{1}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2}$$

where $c = \frac{1}{\sqrt{\epsilon\mu}}$ is the speed of the electromagnetic wave.

In free space $\mu = \mu_0 = 4\pi \times 10^{-7}$ henries/m and $\epsilon = \epsilon_0 \approx \frac{10^{-9}}{36\pi}$ farads/m, which leads to $c_0 = \frac{1}{\sqrt{\epsilon_0\mu_0}} \approx 3 \times 10^8$ m/s.

Similarly

$$\nabla \times (\nabla \times \bar{H}) = \nabla \left( \cancelto{0}{\nabla \cdot \bar{H}} \right) - \nabla^2 \bar{H} = \epsilon \frac{\partial}{\partial t} (\nabla \times \bar{E}) = -\epsilon\mu \frac{\partial^2 \bar{H}}{\partial t^2}$$

$$\nabla^2 \bar{H} = \frac{1}{c^2} \frac{\partial^2 \bar{H}}{\partial t^2}, c = \frac{1}{\sqrt{\epsilon\mu}}$$

B. Plane waves

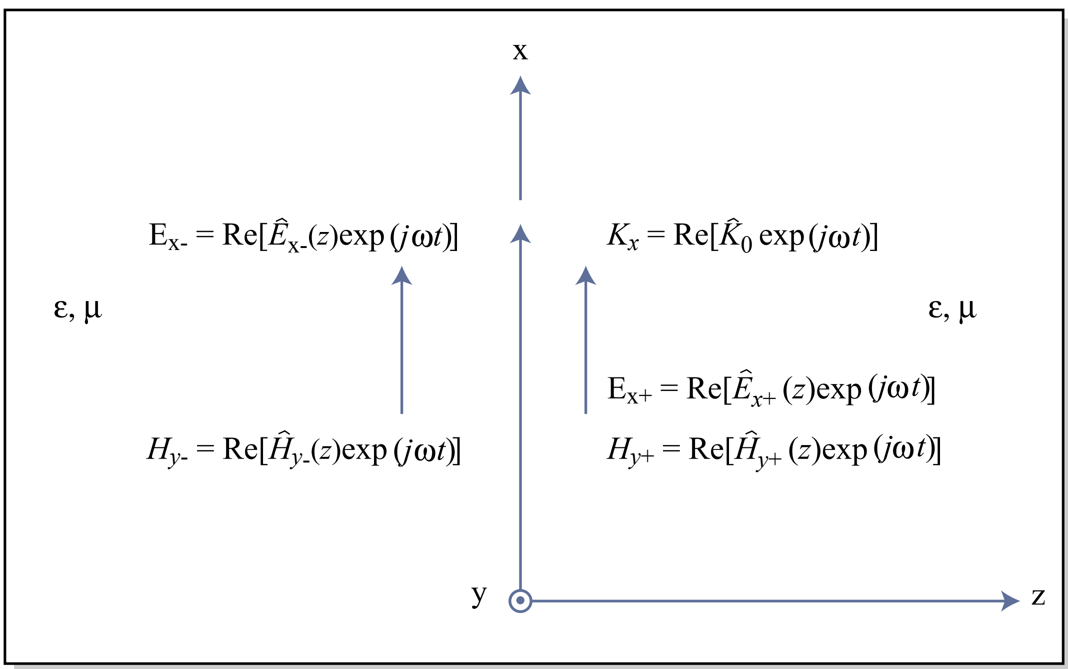


Image by MIT OpenCourseWare.

$$E_x(z, t) = \mathrm{Re}\left[\hat{E}_x(z) e^{j\omega t}\right]$$

$$\frac{d^2 \hat{E}_x}{dz^2} = -\frac{\omega^2}{c^2} \hat{E}_x$$

$$\frac{d^2 \hat{E}_x}{dz^2} + k^2 \hat{E}_x = 0$$

where we have

$$k^2 = \frac{\omega^2}{c^2} = \omega^2 \epsilon\mu$$
$$k = \frac{2\pi}{\lambda} \text{ is the wavenumber}, \lambda \text{ is the wavelength}$$
$$k = \pm\frac{\omega}{c} \Rightarrow \omega = kc$$
$$\omega = 2\pi f = \frac{2\pi}{\lambda} c \Rightarrow f\lambda = c$$

$$\hat{E}_x = A_1 e^{jkz} + A_2 e^{-jkz}$$
$$E_x = \text{Re}\left[ \underbrace{A_1 e^{j(\omega t + kz)}}_{\substack{\text{traveling wave} \\ \text{in the } -z \text{ direction}}} + \underbrace{A_2 e^{j(\omega t - kz)}}_{\substack{\text{traveling wave} \\ \text{in the } +z \text{ direction}}} \right]$$

For the wave in the $-z$ direction we have:

$$\omega t + kz = \text{ constant}$$
$$\omega dt + kdz = 0$$
$$\frac{dz}{dt} = -\frac{\omega}{k} = -c$$

For the wave in the $+z$ direction we have:

$$\omega t - kz = \text{ constant}$$
$$\omega dt - kdz = 0$$
$$\frac{dz}{dt} = \frac{\omega}{k} = +c$$

$$\hat{E}_x(z) = \begin{cases} \hat{E}_{x+} e^{-jkz} & z > 0 \\ \hat{E}_{x-} e^{+jkz} & z < 0 \end{cases}$$

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \Rightarrow \frac{d\hat{E}_x}{dz} = -j\omega\mu \hat{H}_y \Rightarrow \hat{H}_y = -\frac{1}{j\omega\mu}\frac{d\hat{E}_x}{dz}$$
$$\nabla \times \bar{H} = \epsilon \frac{\partial \bar{E}}{\partial t} \Rightarrow \frac{d\hat{H}_y}{dz} = -j\omega\epsilon \hat{E}_x$$

$$\hat{H}_y = \begin{cases} \frac{k}{\omega\mu} \hat{E}_{x+} e^{-jkz} & z > 0 \\ -\frac{k}{\omega\mu} \hat{E}_{x-} e^{+jkz} & z < 0 \end{cases}$$

$$\frac{k}{\omega\mu} = \frac{\not{\omega}\sqrt{\epsilon\mu}}{\not{\omega}\mu} = \sqrt{\frac{\epsilon}{\mu}}, \eta = \sqrt{\frac{\mu}{\epsilon}} \text{ is the wave impedance}$$

$$\hat{H}_y = \begin{cases} \frac{\hat{E}_{x+}}{\eta} e^{-jkz} & z > 0 \\ -\frac{\hat{E}_{x-}}{\eta} e^{+jkz} & z < 0 \end{cases}$$

Now we look at the boundary conditions:

$$E_x(z=0_+,t) = E_x(z=0_-,t) \Rightarrow \hat{E}_{x+} = \hat{E}_{x-}$$

$$H_y(z=0_-,t) - H_y(z=0_+,t) = K_x(z=0,t) \Rightarrow \frac{-\hat{E}_{x-} - \hat{E}_{x+}}{\eta} = \hat{K}_0$$

$$\hat{E}_{x+} = \hat{E}_{x-} = -\frac{\eta\hat{K}_0}{2}$$

$$\hat{E}_x(z) = \begin{cases} -\frac{\eta\hat{K}_0}{2}e^{-jkz} & z>0 \\ -\frac{\eta\hat{K}_0}{2}e^{jkz} & z<0 \end{cases}$$

$$\hat{H}_y(z) = \begin{cases} -\frac{\hat{K}_0}{2}e^{-jkz} & z>0 \\ \frac{\hat{K}_0}{2}e^{jkz} & z<0 \end{cases}$$

$$\hat{S} = \frac{1}{2}\hat{\bar{E}} \times \hat{\bar{H}}^* = \begin{cases} \frac{\eta K_0^2}{8}\bar{i}_z & z>0 \\ -\frac{\eta K_0^2}{8}\bar{i}_z & z<0 \end{cases} \quad (\hat{K}_0 \text{ real})$$

$$E_x(z,t) = \text{Re}\left[\hat{E}_x(z)e^{j\omega t}\right] = \begin{cases} -\frac{\eta K_0}{2}\cos(\omega t - kz) & z>0 \\ -\frac{\eta K_0}{2}\cos(\omega t + kz) & z<0 \end{cases}$$

$$H_y(z,t) = \text{Re}\left[\hat{H}_y(z)e^{j\omega t}\right] = \begin{cases} -\frac{K_0}{2}\cos(\omega t - kz) & z>0 \\ +\frac{K_0}{2}\cos(\omega t + kz) & z<0 \end{cases}$$

$$S_z = E_x H_y = \begin{cases} \frac{\eta K_0^2}{4}\cos^2(\omega t - kz) & z>0 \\ -\frac{\eta K_0^2}{4}\cos^2(\omega t + kz) & z<0 \end{cases}$$

$$\langle S_z \rangle = \begin{cases} \frac{\eta K_0^2}{8} & z>0 \\ -\frac{\eta K_0^2}{8} & z<0 \end{cases}$$

C. Normal Incidence Onto a Perfect Conductor

Incident Fields:

$$\bar{E}_i(z,t) = \text{Re}\left[\hat{E}_i e^{j(\omega t - kz)}\bar{i}_x\right]$$

$$\bar{H}_i(z,t) = \text{Re}\left[\frac{\hat{E}_i}{\eta} e^{j(\omega t - kz)}\bar{i}_y\right]$$

Reflected Fields:

$$\bar{E}_r(z,t) = \text{Re}\left[\hat{E}_r e^{j(\omega t + kz)}\bar{i}_x\right]$$

$$\bar{H}_r(z,t) = \text{Re}\left[-\frac{\hat{E}_r}{\eta} e^{j(\omega t + kz)}\bar{i}_y\right]$$

$$k = \omega\sqrt{\epsilon\mu}, \eta = \sqrt{\frac{\mu}{\epsilon}}$$

The boundary conditions require that

$$E_x(z=0,t) = E_{x,i}(z=0,t) + E_{x,r}(z=0,t) = 0$$

$$\hat{E}_i + \hat{E}_r = 0 \Rightarrow \hat{E}_r = -\hat{E}_i$$

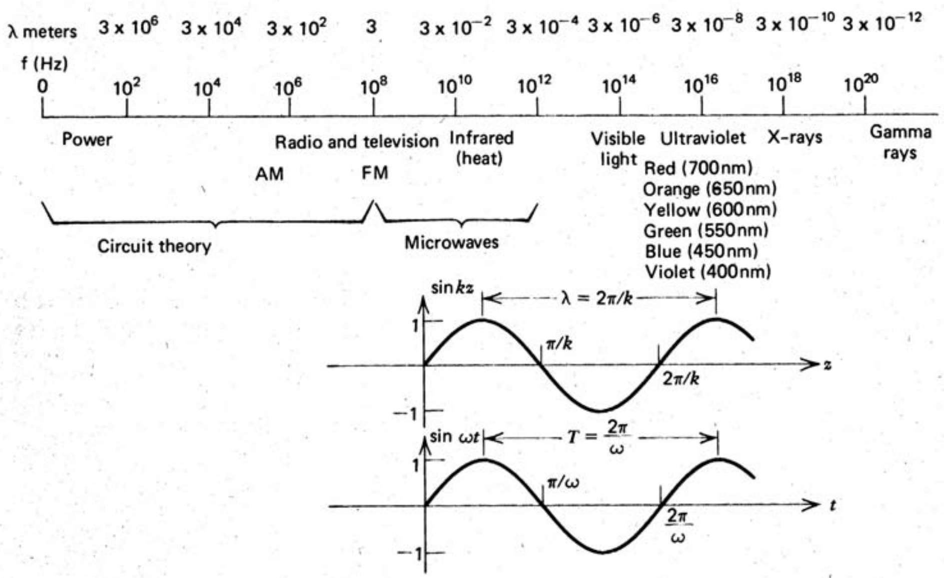


**Figure 7-7** Time varying electromagnetic phenomena differ only in the scaling of time (frequency) and size (wavelength). In linear dielectric media the frequency and wavelength are related as $f\lambda = c$ ($\omega = kc$), where $c = 1/\sqrt{\varepsilon\mu}$ is the speed of light in the medium.



For $\hat{E}_i = E_i$ real we have:

$$\begin{aligned} E_x(z,t) = E_{x,i}(z,t) + E_{x,r}(z,t) &= \mathrm{Re}\left[\hat{E}_i\left(e^{-jkz} - e^{+jkz}\right)e^{j\omega t}\right] \\ &= 2E_i \sin(kz)\sin(\omega t) \\ H_y(z,t) = H_{y,i}(z,t) + H_{y,r}(z,t) &= \mathrm{Re}\left[\frac{\hat{E}_i}{\eta}\left(e^{-jkz} + e^{+jkz}\right)e^{j\omega t}\right] \\ &= \frac{2E_i}{\eta}\cos(kz)\cos(\omega t) \end{aligned}$$

$$K_z(z=0,t) = H_y(z=0,t) = \frac{2E_i}{\eta}\cos(\omega t)$$

Radiation pressure in free space ($\mu = \mu_0, \epsilon = \epsilon_0$)

$$\begin{aligned} \left.\frac{\text{Force}_z}{\text{Area}}\right|_{z=0} = \frac{1}{2}\bar{K}\times\mu_0\bar{H} = \frac{1}{2}\mu_0 K_x H_y|_{z=0}\,\bar{i}_z &= \frac{1}{2}\mu_0 H_y^2(z=0)\bar{i}_z \\ &= \frac{2\mu_0 E_i^2}{\eta_0^2}\cos^2(\omega t)\bar{i}_z \\ &= \frac{2\mu_0}{\mu_0/\epsilon_0}E_i^2\cos^2(\omega t)\bar{i}_z \\ &= 2\epsilon_0 E_i^2\cos^2(\omega t)\bar{i}_z \end{aligned}$$

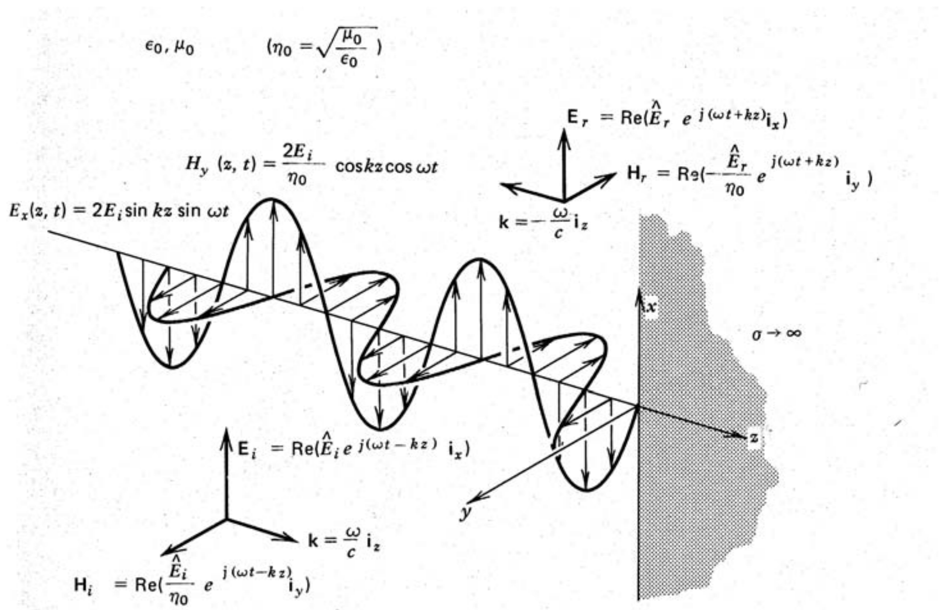


**Figure 7-13** A uniform plane wave normally incident upon a perfect conductor has zero electric field at the conducting surface thus requiring a reflected wave. The source of this reflected wave is the surface current at $z = 0$, which equals the magnetic field there. The total electric and magnetic fields are 90° out of phase in time and space.



IV. Normal Incidence Onto a Dielectric

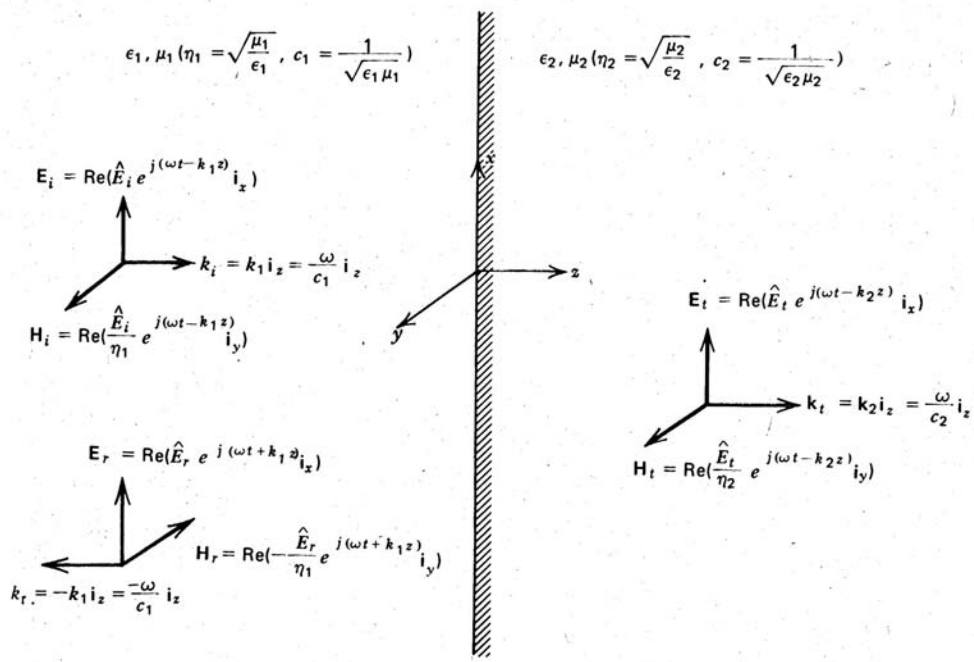


**Figure 7-14** A uniform plane wave normally incident upon a dielectric interface separating two different materials has part of its power reflected and part transmitted.

$$\bar{E}_i(z,t) = \text{Re}\left[\hat{E}_i e^{j(\omega t - k_1 z)}\bar{i}_x\right], k_1 = \omega\sqrt{\epsilon_1\mu_1}$$

$$\bar{H}_i(z,t) = \text{Re}\left[\frac{\hat{E}_i}{\eta_1} e^{j(\omega t - k_1 z)}\bar{i}_y\right], \eta_1 = \sqrt{\mu_1/\epsilon_1}$$

$$\bar{E}_r(z,t) = \text{Re}\left[\hat{E}_r e^{j(\omega t + k_1 z)}\bar{i}_x\right]$$

$$\bar{H}_r(z,t) = \text{Re}\left[\frac{-\hat{E}_r}{\eta_1} e^{j(\omega t + k_1 z)}\bar{i}_y\right]$$

$$\bar{E}_t(z,t) = \text{Re}\left[\hat{E}_t e^{j(\omega t - k_2 z)}\bar{i}_x\right], k_2 = \omega\sqrt{\epsilon_2\mu_2}$$

$$\bar{H}_t(z,t) = \text{Re}\left[\frac{\hat{E}_t}{\eta_2} e^{j(\omega t - k_2 z)}\bar{i}_y\right], \eta_2 = \sqrt{\mu_2/\epsilon_2}$$

$$E_x(z=0_-) = E_x(z=0_+) \Rightarrow \hat{E}_i + \hat{E}_r = \hat{E}_t$$

$$H_y(z=0_-) = H_y(z=0_+) \Rightarrow \frac{\hat{E}_i - \hat{E}_r}{\eta_1} = \frac{\hat{E}_t}{\eta_2}$$

$$R \equiv \frac{\hat{E}_r}{\hat{E}_i} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} \text{ is the Reflection coefficient}$$

$$T \equiv \frac{\hat{E}_t}{\hat{E}_i} = \frac{2\eta_2}{\eta_1 + \eta_2} \text{ is the Transmission coefficient}$$

$$1 + R = T$$

$$\begin{aligned}
\langle S_{z,i}\rangle &= \frac{1}{2}\text{Re}\left[\hat{E}_x(z)\hat{H}_y^*(z)\right] \\
&= \frac{1}{2\eta_1}\text{Re}\left[\left(\hat{E}_i e^{-jk_1 z} + \hat{E}_r e^{jk_1 z}\right)\left(\hat{E}_i^* e^{+jk_1 z} - \hat{E}_r^* e^{-jk_1 z}\right)\right] \\
&= \frac{1}{2\eta_1}\left[|\hat{E}_i|^2 - |\hat{E}_r|^2\right] \\
&\quad + \frac{1}{2\eta_1}\text{Re}\underbrace{\left[\hat{E}_r\hat{E}_i^* e^{2jk_1 z} - \hat{E}_r^*\hat{E}_i e^{-2jk_1 z}\right]}_{\text{pure imaginary}} \\
&= \frac{1}{2\eta_1}\left[|\hat{E}_i|^2 - |\hat{E}_r|^2\right] \\
&= \frac{|\hat{E}_i|^2}{2\eta_1}\left[1 - R^2\right]
\end{aligned}$$

$$\langle S_{z,t}\rangle = \frac{1}{2\eta_2}|\hat{E}_t|^2 = \frac{|\hat{E}_i|^2 T^2}{2\eta_2} = \frac{|\hat{E}_i|^2(1-R^2)}{2\eta_1} = \langle S_{z,i}\rangle$$

V. Lossy Dielectrics - $\bar{J} = \sigma\bar{E}$

Ampere's Law: $\nabla \times \bar{H} = \bar{J} + \epsilon\frac{\partial \bar{E}}{\partial t} = \sigma\bar{E} + \epsilon\frac{\partial \bar{E}}{\partial t}$

For $e^{j\omega t}$ fields

$$\nabla \times \hat{\bar{H}} = (j\omega\epsilon + \sigma)\hat{\bar{E}} = j\omega\epsilon\left(1 + \frac{\sigma}{j\omega\epsilon}\right)\hat{\bar{E}}$$

Define complex permittivity by $\hat{\epsilon} = \epsilon\left(1 + \frac{\sigma}{j\omega\epsilon}\right)$. Then complex amplitude solutions are the same as real amplitude solutions if we replace $\epsilon$ by $\hat{\epsilon}$:

$$k = \omega\sqrt{\hat{\epsilon}\mu}, \qquad \eta = \sqrt{\frac{\mu}{\hat{\epsilon}}} = \sqrt{\frac{\mu}{\epsilon\left(1+\frac{\sigma}{j\omega\epsilon}\right)}}$$

$$k = \omega\sqrt{\epsilon\mu\left(1 + \frac{\sigma}{j\omega\epsilon}\right)}$$

A. Low loss limit: $\frac{\sigma}{\omega\epsilon} \ll 1$

$$\begin{aligned}
k &= \omega\sqrt{\epsilon\mu}\sqrt{1 + \frac{\sigma}{j\omega\epsilon}} \\
&\approx \omega\sqrt{\epsilon\mu}\left(1 + \frac{1}{2}\frac{\sigma}{j\omega\epsilon}\right) \\
&\approx \omega\sqrt{\epsilon\mu} - \frac{j\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \\
&\approx \omega\sqrt{\epsilon\mu} - j\frac{\sigma\eta}{2} \\
e^{-jkz} &= e^{-j\omega\sqrt{\epsilon\mu}z} e^{-j(-j\frac{\sigma\eta}{2})z} \\
&= e^{-j\omega\sqrt{\epsilon\mu}z} \underbrace{e^{-\frac{\sigma\eta}{2}z}}_{\text{slow exponential decay}}
\end{aligned}$$

B. Large loss limit: $\frac{\sigma}{\omega\epsilon} \gg 1$

$$\begin{aligned}
k &= \omega\sqrt{\epsilon\mu}\sqrt{1 + \frac{\sigma}{j\omega\epsilon}} \\
&\approx \omega\sqrt{\not{\epsilon}\mu}\sqrt{\frac{\sigma}{\omega\not{\epsilon}}}\underbrace{\sqrt{-j}}_{\frac{1-j}{\sqrt{2}}} \\
&\approx \sqrt{\frac{\omega\mu\sigma}{2}}(1-j) \\
&\approx \frac{1-j}{\delta} \qquad \delta \equiv \sqrt{\frac{2}{\omega\mu\sigma}} \text{ is the skin depth} \\
e^{-jkz} &= e^{-j(1-j)z/\delta} = e^{-jz/\delta}\underbrace{e^{-z/\delta}}_{\text{fast exponential decay}}
\end{aligned}$$