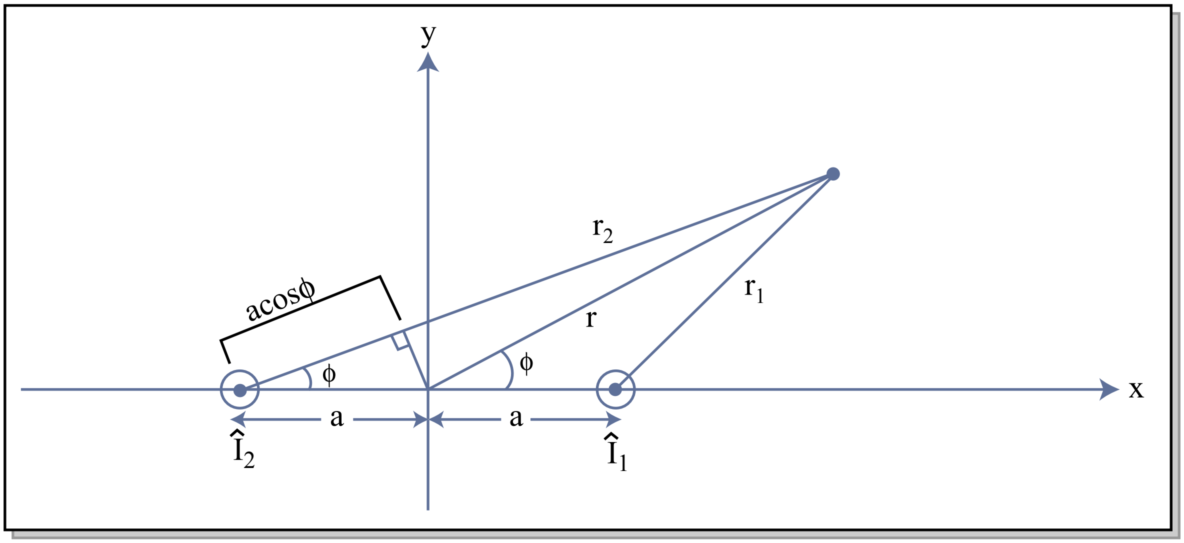
6.013 - Electromagnetics and Applications Fall 2005

Lecture 20 - Dipole Arrays

Prof. Markus Zahn December 1, 2005

I. Two Element Array in $\theta = \frac{\pi}{2}$ plane (x-y plane)



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Far field ($kr \gg 1, r \gg a$)

$$\hat{E}_\theta(r, \theta = \frac{\pi}{2}, \phi) = \frac{\hat{E}_1}{jkr_1} e^{-jkr_1} + \frac{\hat{E}_2}{jkr_2} e^{-jkr_2} = \eta \hat{H}_\phi(r, \theta = \frac{\pi}{2}, \phi)$$

$$\hat{E}_1 = -\frac{\hat{I}_1 dl k^2 \eta}{4\pi}$$

$$\hat{E}_2 = -\frac{\hat{I}_2 dl k^2 \eta}{4\pi}$$

$$r_2 \approx r + a\cos(\phi), r_1 \approx r - a\cos(\phi)$$

$$\hat{E}_\theta(r, \theta = \frac{\pi}{2}, \phi) = \eta \hat{H}_\phi(r, \theta = \frac{\pi}{2}, \phi) \approx \underbrace{-\frac{k^{\not 2} \eta dl}{4\pi j \not{k} r} e^{-jkr}}_{\text{element factor}} \left[\underbrace{\hat{I}_1 e^{+jka\cos(\phi)} + \hat{I}_2 e^{-jka\cos(\phi)}}_{\text{array factor}} \right]$$

Assume: $\hat{I}_1 = \hat{I}, \hat{I}_2 = \hat{I} e^{j\chi} \Rightarrow \hat{E}_1 = \hat{E}_0, \hat{E}_2 = \hat{E}_0 e^{j\chi}$

$$E_\theta(r, \theta = \frac{\pi}{2}, \phi) = \eta \hat{H}_\phi(r, \theta = \frac{\pi}{2}, \phi) = \frac{\hat{E}_0}{jkr} e^{-jkr} \left[e^{+jka\cos(\phi)} + e^{j\chi} e^{-jka\cos(\phi)} \right]$$

$$= \frac{\hat{E}_0}{jkr} e^{-jkr} e^{j\chi/2} \left[\underbrace{e^{-j(\frac{\chi}{2} - ka\cos(\phi))} + e^{j(\frac{\chi}{2} - ka\cos(\phi))}}_{2\cos\left(-\frac{\chi}{2} + ka\cos(\phi)\right)} \right]$$

$$= \frac{2\hat{E}_0}{jkr} e^{-jkr} e^{j\chi/2} \cos\left(-\frac{\chi}{2} + ka\cos(\phi)\right)$$

$$\left\langle S_r(t, \theta = \frac{\pi}{2}, \phi) \right\rangle = \frac{1}{2} \frac{|\hat{E}_\theta|^2}{\eta} = \frac{2|\hat{E}_0|^2}{\eta (kr)^2} \cos^2\left(ka\cos(\phi) - \frac{\chi}{2}\right)$$

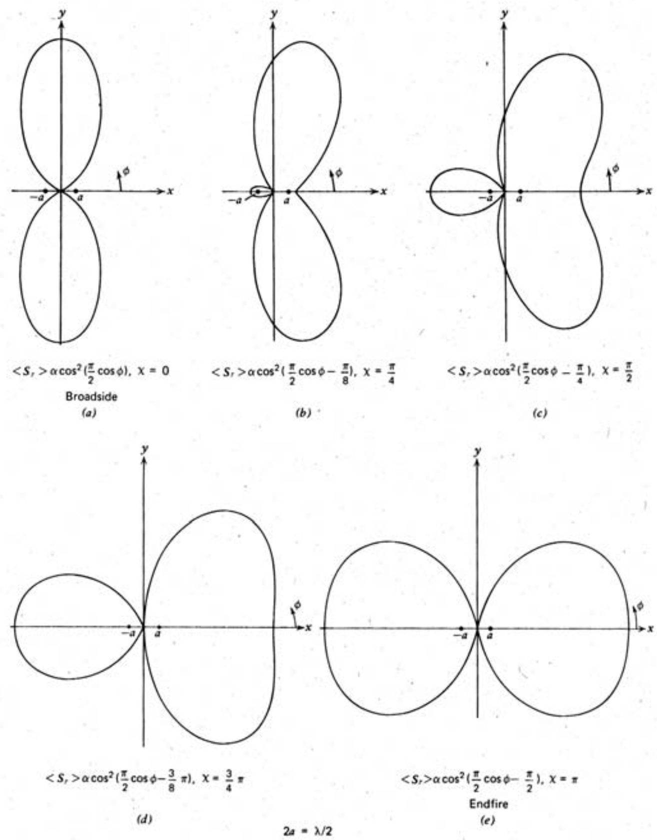


Figure 9-7 The power radiation pattern due to two-point dipoles depends strongly on the dipole spacing and current phases. With a half wavelength dipole spacing ($2a = \lambda/2$), the radiation pattern is drawn for various values of current phase difference in the $\theta = \pi/2$ plane. The broadside array in (*a*) with the currents in phase ($\chi = 0$) has the power lobe in the direction perpendicular to the array while the end-fire array in (*e*) has out-of-phase currents ($\chi = \pi$) with the power lobe in the direction along the array.

Broadside:

$$2a = \frac{\lambda}{2}, \chi = 0, ka = \frac{\pi}{2}$$
$$\langle S_r \rangle = \frac{2|\hat{E}_0|^2}{\eta (kr)^2} \cos^2\left(\frac{\pi}{2}\cos(\phi)\right)$$

Endfire:

$$2a = \frac{\lambda}{2}, \chi = \pi, ka = \frac{\pi}{2}$$
$$\langle S_r \rangle = \frac{2|\hat{E}_0|^2}{\eta (kr)^2} \cos^2\left(\frac{\pi}{2}\left(\cos(\phi) - 1\right)\right)$$

Maxima: $ka\cos(\phi) - \frac{\chi}{2} = \pm m\pi, m = 0, 1, 2, \ldots$

Minima: $ka\cos(\phi) - \frac{\chi}{2} = \pm(2m+1)\frac{\pi}{2}, m = 0, 1, 2, \ldots$

Case Studies:

$$2a = \frac{\lambda}{2} \Rightarrow ka = \frac{2\pi a}{\lambda} = \frac{2\pi a}{4a} = \frac{\pi}{2}$$
$$2a = \lambda \Rightarrow ka = \pi$$

$$2a = \frac{\lambda}{2} \Rightarrow \frac{\pi}{2}\cos(\phi) - \frac{\chi}{2} = \pm m\pi \text{ (maxima)} \Rightarrow \cos(\phi) = \frac{\chi}{\pi} \pm 2m$$
$$\frac{\pi}{2}\cos(\phi) - \frac{\chi}{2} = \pm(2m+1)\frac{\pi}{2} \text{ (minima)} \Rightarrow \cos(\phi) = \frac{\chi}{\pi} \pm (2m+1)$$

$\lambda = 4a$

χ	$\cos(\phi_{\max})$	$\cos(\phi_{\min})$	$\phi_{\max}$	$\phi_{\min}$	
0	0	1	$\pm\frac{\pi}{2}$	$0, \pi$	Broadside
$\frac{\pi}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	$\pm 75.5°$	$\pm 138.6°$	
$\frac{\pi}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\pm 60°$	$\pm 120°$	
$\frac{3\pi}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$\pm 41.4°$	$\pm 104.5°$	
π	1	0	$0, \pi$	$\pm 90.°$	Endfire

$\lambda = 2a \Rightarrow \cos(\phi_{\max}) = \frac{\chi}{2\pi} \pm m, \cos(\phi_{\min}) = \frac{\chi}{2\pi} \pm \frac{1}{2}(2m+1)$

χ	$\cos(\phi_{\max})$	$\cos(\phi_{\min})$	$\phi_{\max}$	$\phi_{\min}$
0	$0, 1$	$\frac{1}{2}, -\frac{1}{2}$	$0, \pm 90°, 180°$	$\pm 60°$
$\frac{\pi}{4}$	$\frac{1}{8}, -\frac{7}{8}$	$-\frac{3}{8}, \frac{5}{8}$	$82.8°, 151°$	$51°, 112°$
$\frac{\pi}{2}$	$\frac{1}{4}, -\frac{3}{4}$	$-\frac{1}{4}, \frac{3}{4}$	$75.5°, 138.6°$	$41.4°, 104.5°$
$\frac{3\pi}{4}$	$\frac{3}{8}, -\frac{5}{8}$	$-\frac{1}{8}, \frac{7}{8}$	$68.0°, 128.7°$	$29.0°, 97.2°$
π	$\frac{1}{2}, -\frac{1}{2}$	$0, 1$	$60°, 120°$	$90°, 0°$

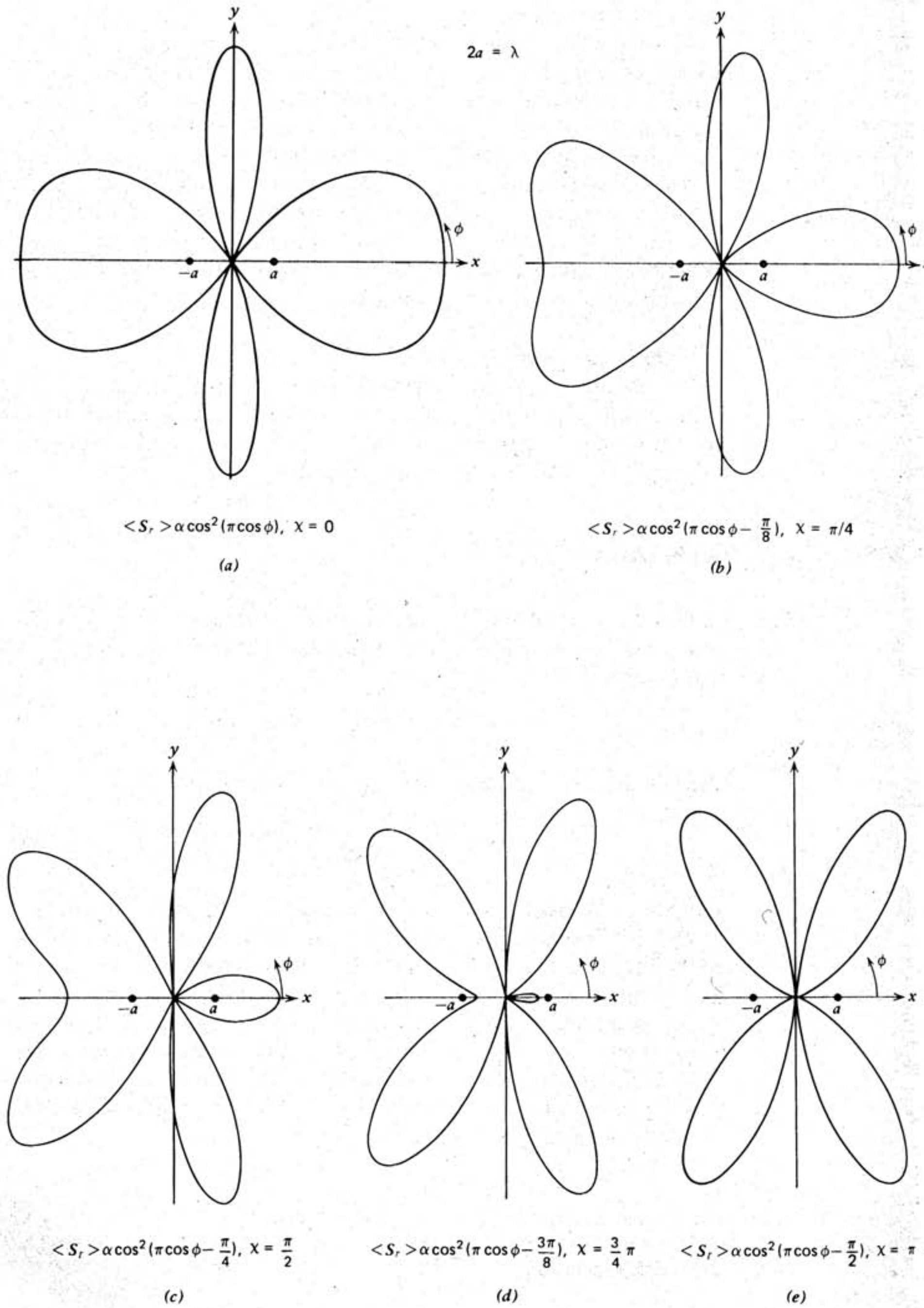


Figure 9-8 With a full wavelength dipole spacing ($2a = \lambda$) there are four main power lobes.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

II. An N Dipole Array ($\theta = \frac{\pi}{2}$)

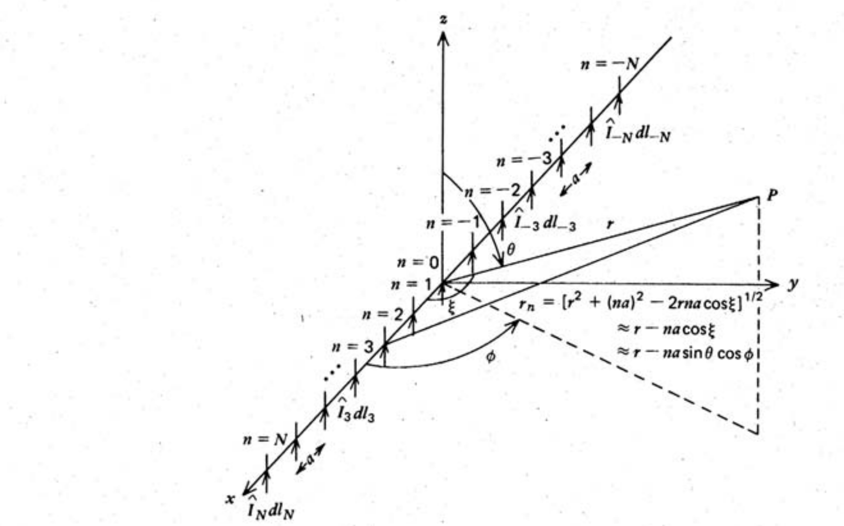


Figure 9-9 **A linear point dipole array with $2N+1$ equally spaced dipoles.**

$$\lim_{r \gg na} r_n \approx r - na\cos(\phi)$$
$$-N \le n \le N$$

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$\hat{E}_\theta\left(r, \theta = \frac{\pi}{2}, \phi\right) = \eta \hat{H}_\phi\left(r, \theta = \frac{\pi}{2}, \phi\right)$$
$$= -\frac{k\eta dl}{4\pi j r} \underbrace{\left[\sum_{-N}^{+N} \hat{I}_n e^{jkna\cos(\phi)}\right]}_{\text{Array factor = AF}} e^{-jkr}$$

Example:

$$\hat{I}_n = I_0 e^{-jn\chi_0} \qquad -N \le n \le N$$
$$\text{AF} = I_0 \sum_{-N}^{+N} e^{jn(ka\cos(\phi)-\chi_0)}$$
$$\text{Let } \beta \equiv e^{j(ka\cos(\phi)-\chi_0)}$$
$$S = \frac{\text{AF}}{I_0} = \sum_{-N}^{+N} \beta^n = \beta^{-N} + \beta^{-N+1} + \ldots + \beta^{-2} + \beta^{-1} + 1 + \beta + \beta^2 + \ldots + \beta^{N-1} + \beta^N$$

$$S(1-\beta) = \beta^{-N} - \beta^{N+1} \Rightarrow S = \underbrace{\frac{\beta^{-N} - \beta^{N+1}}{1-\beta}}_{\substack{\text{multiply by} \\ \frac{\beta^{-1/2}}{\beta^{-1/2}}}} = \frac{\beta^{-N-\frac{1}{2}} - \beta^{N+\frac{1}{2}}}{\beta^{-1/2} - \beta^{1/2}}$$

$$S = \frac{\sin\left[(N+\frac{1}{2})(ka\cos(\phi) - \chi_0)\right]}{\sin\left[\frac{1}{2}(ka\cos(\phi) - \chi_0)\right]}$$

Maxima: $ka\cos(\phi) - \chi_0 = 2n\pi, n = 0, 1, 2, \ldots$

Principal maximum at $n = 0 \Rightarrow \cos(\phi) = \frac{\chi_0}{ka}$

Minima: $(N + \frac{1}{2})(ka\cos(\phi) - \chi_0) = n\pi, n = 1, 2, 3, \ldots$

Demonstration, $N = 2$ (2 dipole array)

$$2a = \frac{3}{2}\lambda, \chi_0 = 0$$

$$I \propto \cos^2(ka\cos(\phi)) = \cos^2\left(\frac{\cancel{2}\pi}{\cancel{\lambda}}\frac{3}{\cancel{4}_2}\cancel{\lambda}\cos(\phi)\right) = \cos^2\left(\frac{3\pi}{2}\cos(\phi)\right)$$

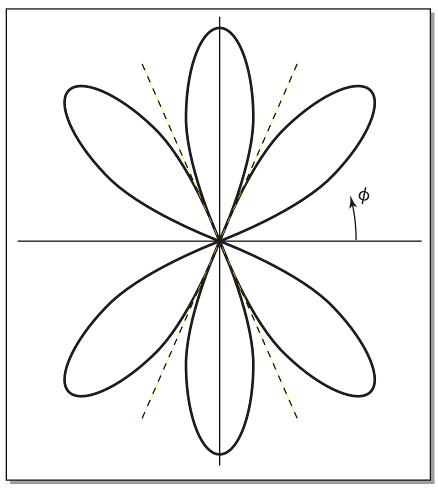
$$\text{Minima: } \frac{3\cancel{\pi}}{\cancel{2}}\cos(\phi) = \frac{\cancel{\pi}}{\cancel{2}} \Rightarrow \cos(\phi) = \frac{1}{3} \Rightarrow \phi = 70.5^\circ$$

$$\frac{\cancel{3\pi}}{\cancel{2}}\cos(\phi) = \frac{\cancel{3\pi}}{\cancel{2}} \Rightarrow \cos(\phi) = 1 \Rightarrow \phi = 0^\circ$$

$$\text{Maxima: } \frac{3\pi}{2}\cos(\phi) = 0 \Rightarrow \phi = 90^\circ$$

$$\frac{3\cancel{\pi}}{2}\cos(\phi) = \cancel{\pi} \Rightarrow \cos(\phi) = \frac{2}{3} \Rightarrow \phi = 48.2^\circ$$

Image by MIT OpenCourseWare.



Intensity pattern

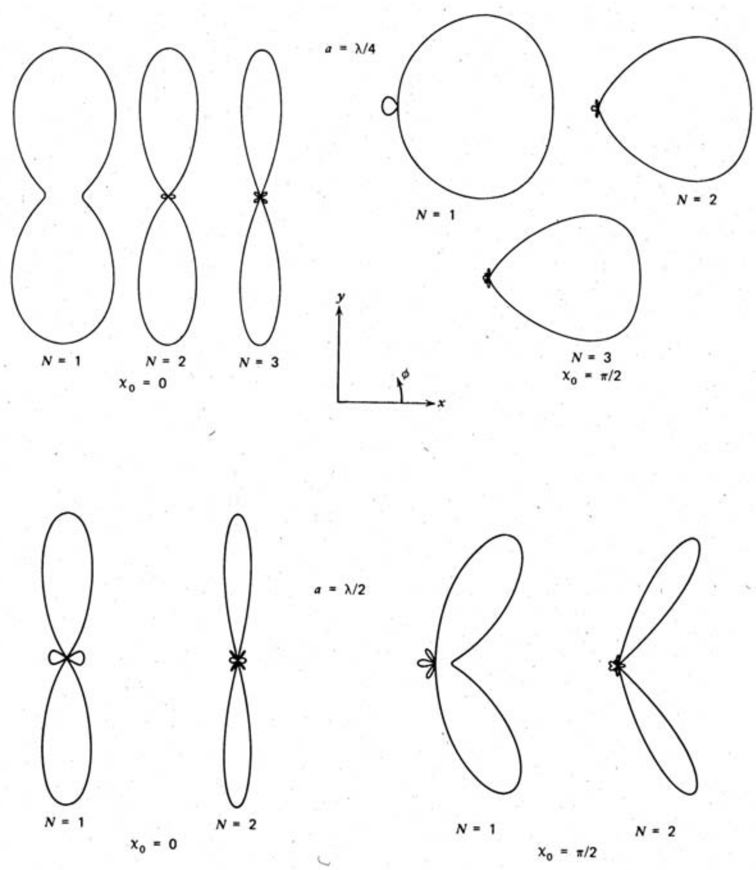


Figure 9-10 The radiation pattern for an N dipole linear array for various values of N, dipole spacing $2a$, and relative current phase χ_0 in the $\theta = \pi/2$ plane.

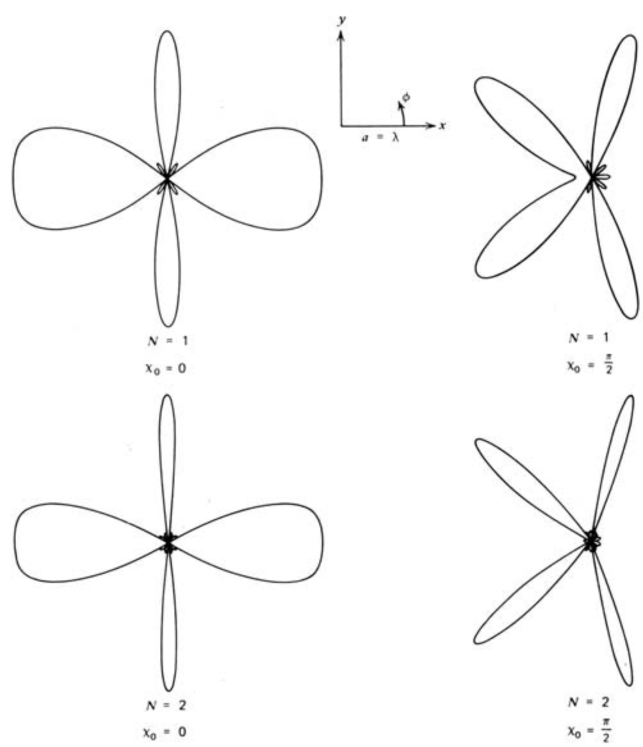


Figure 9-10