6.013 - Electromagnetics and Applications	Fall 2005
Lecture 19 - Electromagnetic Radiation	
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Lecture 19 - Electromagnetic Radiation

I. Retarded Potentials

A. Maxwell's Equations

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$
$$\nabla \times \bar{H} = \bar{J}_f + \frac{\partial \bar{D}}{\partial t}$$
$$\nabla \cdot \bar{B} = 0$$
$$\nabla \cdot \bar{D} = \rho_f$$
$$\bar{B} = \mu \bar{H}, \bar{D} = \epsilon \bar{E}$$

B. Vector Identities

$$\nabla \times (\nabla \Phi) = 0$$
$$\nabla \cdot (\nabla \times \bar{A}) = 0$$
$$\nabla \times (\nabla \times \bar{A}) = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$$

C. Potentials

$$\nabla \cdot \bar{B} = 0 \Rightarrow \bar{B} = \nabla \times \bar{A}$$
$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \Rightarrow \nabla \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t}\right) = 0$$
$$\Rightarrow \bar{E} + \frac{\partial \bar{A}}{\partial t} = -\nabla \Phi$$
$$\nabla \times \bar{H} = \bar{J}_f + \epsilon \frac{\partial \bar{E}}{\partial t} \Rightarrow \frac{1}{\mu} \nabla \times (\nabla \times \bar{A}) = \bar{J}_f + \epsilon \frac{\partial}{\partial t}\left(-\frac{\partial \bar{A}}{\partial t} - \nabla \Phi\right)$$
$$\nabla \times (\nabla \times \bar{A}) = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \mu \bar{J}_f - \epsilon\mu \frac{\partial^2 \bar{A}}{\partial t^2} - \epsilon\mu \nabla \frac{\partial \Phi}{\partial t}$$
$$\nabla^2 \bar{A} - \nabla\left[\nabla \cdot \bar{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right] - \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} = -\mu \bar{J}_f, c^2 = \frac{1}{\epsilon\mu}$$

Setting the gauge (Lorentz gauge):

$$\nabla \cdot \bar{A} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial t}$$
$$\nabla^2 \bar{A} - \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} = -\mu \bar{J}_f$$
$$\nabla \cdot \bar{E} = -\frac{\partial}{\partial t}\left(\nabla \cdot \bar{A}\right) - \nabla^2 \Phi = \frac{\rho_f}{\epsilon} \Rightarrow \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho_f}{\epsilon}$$

D. Solutions to the Wave Equations

1. Solution in Time Domain

   In spherical coordinates:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2\Phi}{\partial\phi^2}$$

   Consider a point charge $Q(t)$ at $r=0$. Then $\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\phi} = 0, \Phi = \Phi(r,t)$:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)$$

   Then

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \text{ for } r>0 \Rightarrow \frac{\partial^2}{\partial r^2}(r\Phi) - \frac{1}{c^2}\frac{\partial^2(r\Phi)}{\partial t^2} = 0$$

   This is a wave equation in $r\Phi(r,t)$

$$r\Phi(r,t) = f_+\left(t-\frac{r}{c}\right) + \underbrace{\cancelto{0}{f_-\left(t+\frac{r}{c}\right)}}_{\substack{\text{No sources for } r>0 \\ \text{so only waves} \\ \text{emanating radially} \\ \text{outward from point} \\ \text{charge at } r=0}}$$

$$\lim_{r\to 0}\Phi(r,t) = \frac{f_+(t)}{r} = \frac{Q(t)}{4\pi\epsilon r} \Rightarrow f_+(t) = \frac{Q(t)}{4\pi\epsilon}$$

$$\Phi(r,t) = \frac{Q\left(t-\frac{r}{c}\right)}{4\pi\epsilon r}$$

2. Solution in Frequency Domain

$$\Phi(r,t) = \text{Re}\left[\hat{\Phi}(r)e^{j\omega t}\right], Q(t) = \text{Re}\left[\hat{Q}e^{j\omega t}\right]$$

$$\frac{d^2}{dr^2}\left(r\hat{\Phi}(r)\right) + \frac{\omega^2}{c^2}r\hat{\Phi}(r) = 0$$

$$r\hat{\Phi}(r) = A_1e^{-jkr} + \underbrace{\cancelto{0}{A_2e^{+jkr}}}_{\substack{\text{No } -r \text{ traveling} \\ \text{wave}}}, \; k = \frac{\omega}{c}$$

$$\lim_{r\to 0}\hat{\Phi}(r) = \frac{A_1}{r} = \frac{\hat{Q}}{4\pi\epsilon r} \Rightarrow A_1 = \frac{\hat{Q}}{4\pi\epsilon}$$

$$\hat{\Phi}(r) = \frac{\hat{Q}}{4\pi\epsilon r}e^{-jkr} \Rightarrow \Phi(r,t) = \text{Re}\Big[\underbrace{\frac{\hat{Q}}{4\pi\epsilon r}e^{j\omega\left(t-\frac{r}{c}\right)}}_{\text{Spherical Wave}}\Big]$$

Vector Potential

$$\nabla \cdot \bar{A} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial t} \Rightarrow \nabla \cdot \hat{\bar{A}} = -\frac{j\omega}{c^2}\hat{\Phi}$$

$$\nabla \cdot \hat{\bar{A}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \hat{A}_r\right) + \cancelto{0}{\frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\hat{A}_\theta\right)} + \cancelto{0}{\frac{1}{r\sin(\theta)}\frac{\partial \hat{A}_\phi}{\partial \phi}}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \phi} = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \hat{A}_r\right) = -\frac{j\omega}{c^2}\hat{\Phi} = -j\omega\cancel{\epsilon}\mu \frac{\hat{Q}}{4\pi\cancel{\epsilon} r}e^{-jkr}$$

$$\frac{\partial}{\partial r}\left(r^2 \hat{A}_r\right) = \frac{d}{dr}\left(r^2 \hat{A}_r\right) = -\frac{j\omega\mu}{4\pi}\hat{Q} r e^{-jkr}$$

$$\begin{aligned} r^2 \hat{A}_r &= -\frac{j\omega\mu}{4\pi}\hat{Q}\int dr\, r e^{-jkr} \\ &= -\frac{j\omega\mu\hat{Q}}{4\pi}\frac{e^{-jkr}}{(+k^2)}(+jkr+1) \\ \hat{A}_r &= -\frac{j\omega\mu\hat{Q}}{4\pi k^2 r^2}(1+jkr)e^{-jkr} \\ &= -\frac{j\cancel{\omega}\cancel{\mu}\hat{Q}}{4\pi\omega^{\cancel{2}}\epsilon\cancel{\mu} r^2}(1+jkr)e^{-jkr} \\ &= -\frac{j\hat{Q}}{4\pi\omega\epsilon r^2}(1+jkr)e^{-jkr} \end{aligned}$$

3. Solutions for Volume Charge and Current Distributions

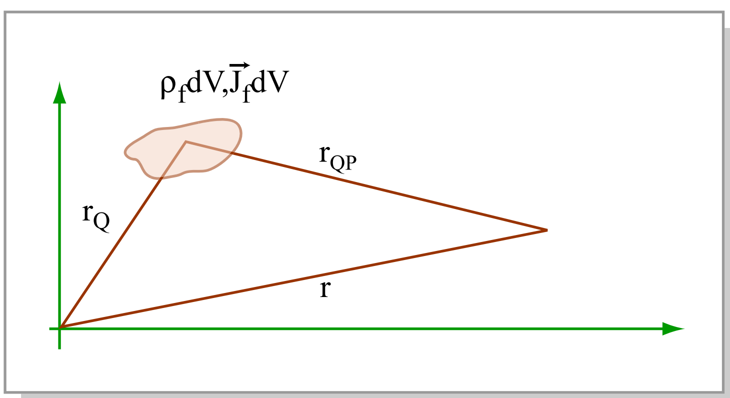

Image by MIT OpenCourseWare.

$$\Phi(r,\theta,\phi,t) = \int_{\text{all charge}} dV\, \frac{\rho_f\left(t - \frac{r_{QP}}{c}\right)}{4\pi\epsilon r_{QP}}$$

$$\nabla^2 \bar{A} - \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} = -\mu \bar{J}_f \Rightarrow \bar{A} = \int_{\text{all current}} dV\, \frac{\mu \bar{J}_f\left(t - \frac{r_{QP}}{c}\right)}{4\pi r_{QP}}$$

II. Radiation From a Point Electric Dipole

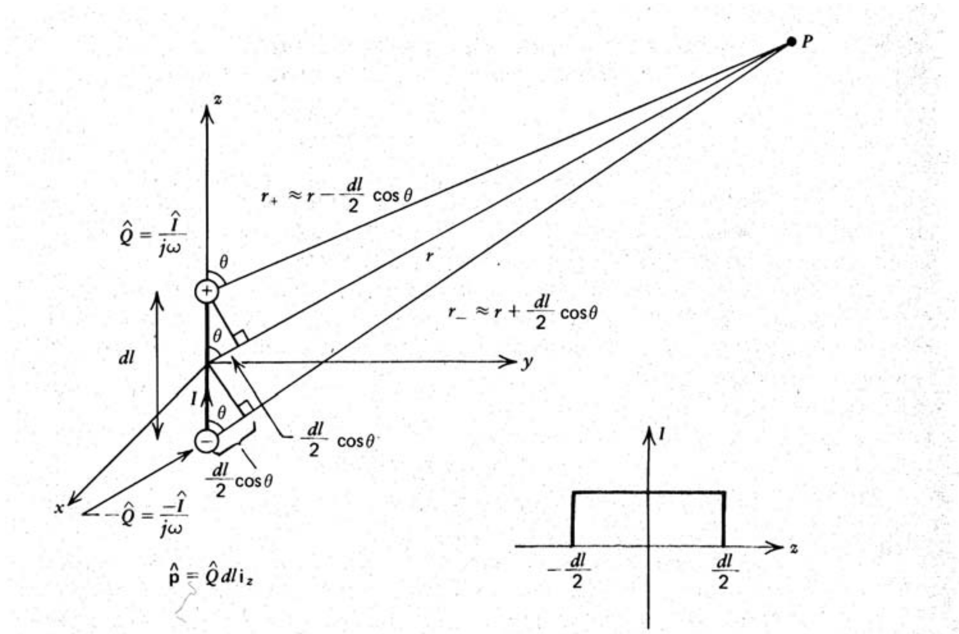

Figure 9-1 A point dipole antenna is composed of a very short uniformly distributed current-carrying wire. Because the current is discontinuous at the ends, equal magnitude but opposite polarity charges accumulate there forming an electric dipole.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$\bar{i}(t) = \text{Re}\left[\hat{I}e^{j\omega t}\,\bar{i}_z\right]$$
$$\hat{Q} = \frac{\hat{I}}{j\omega} \text{ at } z = \pm\frac{dl}{2}$$
$$\hat{\bar{p}} = \hat{Q}dl\bar{i}_z \qquad \text{(dipole moment)}$$

A. Vector Potential

$$A_z(r,\theta,t) = \text{Re}\left[\hat{A}_z(r,\theta)e^{j\omega t}\right], \frac{\partial}{\partial\phi} = 0$$
$$A_z(r,t) = \text{Re}\left[\int_{-\frac{dl}{2}}^{+\frac{dl}{2}} dz\, \frac{\mu\hat{I}e^{j\omega(t-\frac{r_{QP}}{c})}}{4\pi r_{QP}}\right]$$
$$= \text{Re}\left[\frac{\mu\hat{I}}{4\pi r}e^{j\omega(t-\frac{r}{c})}\right]$$
$$\hat{A}_z(r) = \frac{\mu\hat{I}dl}{4\pi r}e^{-jkr}, k = \frac{\omega}{c}$$
$$A_z(r,t) = \text{Re}\left[\hat{A}_z(r)e^{j\omega t}\right]$$
$$\bar{i}_z = \bar{i}_r\cos(\theta) - \bar{i}_\theta\sin(\theta)$$
$$\bar{A} = A_z\bar{i}_z = A_z(\bar{i}_r\cos(\theta) - \bar{i}_\theta\sin(\theta))$$
$$\hat{A}_r = \hat{A}_z\cos(\theta),\ \hat{A}_\theta = -\hat{A}_z\sin(\theta)$$

B. Scalar Potential

$$\hat{\Phi} = \frac{jc^2}{\omega}\nabla\cdot\hat{\bar{A}} = \frac{jc^2}{\omega}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\hat{A}_r\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\hat{A}_\theta\right)\right]$$

$$= \frac{j}{\omega\epsilon\cancel{\mu}}\frac{\cancel{\mu}\hat{I}dl}{4\pi}\left[\frac{\cos(\theta)}{r^2}\frac{d}{dr}\left(\frac{r^{\cancel{2}}e^{-jkr}}{\cancel{r}}\right) - \frac{\cancel{\sin(\theta)}}{r\cancel{\sin(\theta)}}\frac{\partial}{\partial\theta}\left(\frac{\sin(\theta)}{r}e^{-jkr}\right)\right]$$

$$= \frac{-j\hat{I}dl(1+jkr)e^{-jkr}}{4\pi\omega\epsilon r^2}\cos(\theta)$$

$$= \frac{\hat{Q}dl}{4\pi\epsilon r^2}(1+jkr)e^{-jkr}\cos(\theta)$$

C. Magnetic Field

$$\hat{\bar{H}} = \frac{1}{\mu}\nabla\times\hat{\bar{A}} = \frac{1}{\mu}\left[\frac{1}{r\sin(\theta)}\left(\frac{\partial}{\partial\theta}\left(\cancelto{0}{A_\phi}\sin(\theta)\right) - \cancelto{0}{\frac{\partial A_\theta}{\partial\phi}}\right)\bar{i}_r\right.$$

$$\left. + \frac{\bar{i}_\theta}{r}\left(\frac{1}{\sin(\theta)}\cancelto{0}{\frac{\partial A_r}{\partial\phi}} - \frac{\partial}{\partial r}\left(r\cancelto{0}{A_\phi}\right)\right) + \frac{\bar{i}_\phi}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\right]$$

$$\left(\text{with } A_\phi = 0, \frac{\partial}{\partial\phi} = 0\right)$$

$$\bar{H} = \frac{1}{\mu}\nabla\times\bar{A} = \bar{i}_\phi\frac{1}{\mu r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)$$

$$\hat{H}_\phi = -\bar{i}_\phi\frac{\hat{I}dlk^2}{4\pi}\sin(\theta)\left(\frac{1}{jkr} + \frac{1}{(jkr)^2}\right)e^{-jkr}$$

D. Electric Field

$$\hat{\bar{E}} = \frac{1}{j\omega\epsilon}\nabla\times\hat{\bar{H}} = \frac{1}{j\omega\epsilon}\left[\frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\hat{H}_\phi\sin(\theta)\right)\bar{i}_r - \frac{1}{r}\frac{\partial}{\partial r}\left(r\hat{H}_\phi\right)\bar{i}_\theta\right]$$

$$= -\frac{\hat{I}dlk^2}{4\pi}\sqrt{\frac{\mu}{\epsilon}}\left[\bar{i}_r\cdot 2\cos(\theta)\left(\frac{1}{(jkr)^2} + \frac{1}{(jkr)^3}\right)\right.$$

$$\left. + \bar{i}_\theta\sin(\theta)\left(\frac{1}{jkr} + \frac{1}{(jkr)^2} + \frac{1}{(jkr)^3}\right)\right]e^{-jkr}$$

E. Far Field Limit ($kr \gg 1$)

$$\lim_{kr\gg 1}\underbrace{\hat{E}_\theta = \sqrt{\frac{\mu}{\epsilon}}\hat{H}_\phi}_{\text{Same relationship as plane waves}} = \frac{\hat{E}_0}{jkr}\sin(\theta)e^{-jkr}, \hat{E}_0 = -\frac{\hat{I}dlk^2}{4\pi}\sqrt{\frac{\mu}{\epsilon}}$$

III. Power Density

$$
\begin{aligned}
\langle\bar{S}\rangle &= \frac{1}{2}\operatorname{Re}\left(\hat{\bar{E}}\times\hat{\bar{H}}^*\right)\\
\hat{\bar{E}} &= \hat{E}_0\left[2\cos(\theta)\bar{i}_r\left(\frac{1}{(jkr)^2}+\frac{1}{(jkr)^3}\right)+\sin(\theta)\bar{i}_\theta\left(\frac{1}{jkr}+\frac{1}{(jkr)^2}+\frac{1}{(jkr)^3}\right)\right]e^{-jkr}\\
\hat{\bar{H}} &= \frac{\hat{E}_0}{\eta}\bar{i}_\phi\sin(\theta)\left(\frac{1}{jkr}+\frac{1}{(jkr)^2}\right)e^{-jkr}\\
\langle\bar{S}\rangle &= \operatorname{Re}\left\{\frac{|\hat{E}_0|^2}{2\eta}\sin(\theta)\left[2\cos(\theta)\left(\frac{1}{(jkr)^2}+\frac{1}{(jkr)^3}\right)\right]\left[-\frac{1}{jkr}+\frac{1}{(jkr)^2}\right]\underbrace{\bar{i}_r\times\bar{i}_\phi}_{-\bar{i}_\theta}\right.\\
&\qquad\left.+\sin(\theta)\left(\frac{1}{jkr}+\frac{1}{(jkr)^2}+\frac{1}{(jkr)^3}\right)\left(-\frac{1}{jkr}+\frac{1}{(jkr)^2}\right)\underbrace{\bar{i}_\theta\times\bar{i}_\phi}_{\bar{i}_r}\right\}\\
\langle\bar{S}\rangle &= \frac{|\hat{E}_0|^2}{2\eta}\sin(\theta)\operatorname{Re}\left\{-2\cos(\theta)\left(\underbrace{-\frac{1}{(jkr)^3}}_{\text{imag}}\underbrace{-\frac{1}{(jkr)^4}+\frac{1}{(jkr)^4}}_{\text{cancel}}+\underbrace{\frac{1}{(jkr)^5}}_{\text{imag}}\right)\bar{i}_\theta\right.\\
&\qquad\left.+\sin(\theta)\left(\underbrace{-\frac{1}{(jkr)^2}}_{\text{real}}+\underbrace{\frac{1}{(jkr)^3}-\frac{1}{(jkr)^3}}_{\text{imaginary/cancel}}+\underbrace{\frac{1}{(jkr)^4}-\frac{1}{(jkr)^4}}_{\text{cancel}}+\underbrace{\frac{1}{(jkr)^5}}_{\text{imag}}\right)\bar{i}_r\right\}\\
&= \frac{|\hat{E}_0|^2}{2\eta}\frac{\sin^2(\theta)}{k^2r^2}\bar{i}_r,\ \hat{E}_0=-\frac{\hat{I}dlk^2}{4\pi}\eta,\ \eta=\sqrt{\frac{\mu}{\epsilon}}
\end{aligned}
$$

IV. Total Time Average Radiated Power

A. Power

$$
\begin{aligned}
\langle P\rangle &= \int_0^\pi d\theta\int_0^{2\pi}d\phi\,\langle S_r\rangle\,r^2\sin(\theta)\\
&= \frac{|\hat{E}_0|^2}{2\eta k^2}\int_0^\pi d\theta\int_0^{2\pi}d\phi\sin^3(\theta)\\
&= \frac{|\hat{E}_0|^2\,\cancel{2}\pi}{\cancel{2}\eta k^2}\int_0^\pi d\theta\sin^3(\theta)\\
&= \frac{\pi|\hat{E}_0|^2}{\eta k^2}\underbrace{\left[-\frac{1}{3}\cos(\theta)(\sin^2(\theta)+2\right]\Big|_0^\pi}_{\frac{4}{3}}\\
&= \frac{4\pi}{3\eta k^2}|\hat{E}_0|^2\\
&= \frac{\cancel{4\pi}}{3\cancel{\eta}\cancel{k^2}}\frac{|\hat{I}dl|^2k^{\cancel{4}2}}{4\cancel{16}\pi^{\cancel{2}}}\eta^{\cancel{2}}\\
&= \frac{|\hat{I}dl|^2\eta k^2}{12\pi}
\end{aligned}
$$

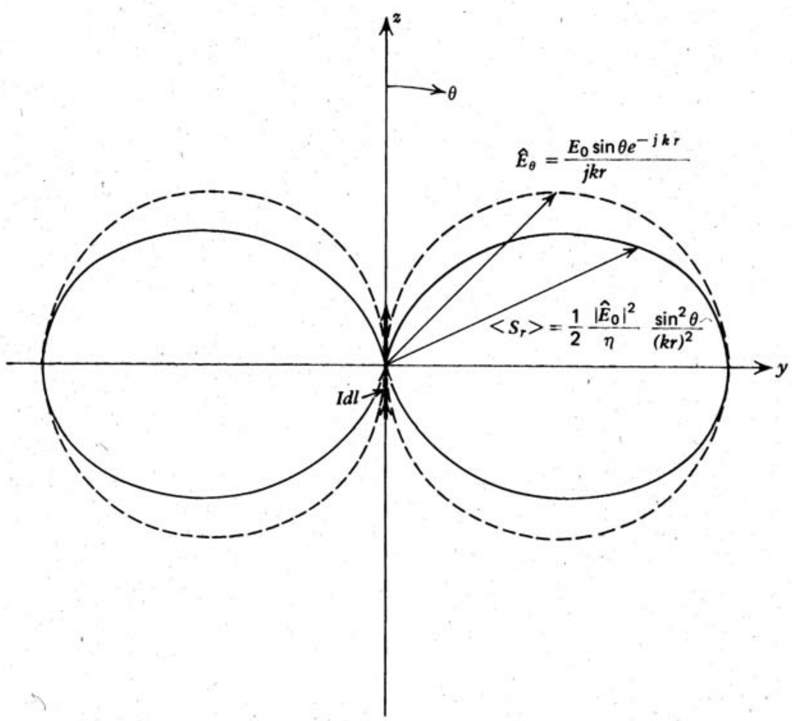

Figure 9-3 The strength of the electric field and power density due to a z-directed point dipole as a function of angle θ is proportional to the length of the vector from the origin to the radiation pattern.

Antenna Gain:

$$G(\theta,\phi) = \frac{\langle S \rangle_r}{\langle P \rangle / 4\pi r^2}$$

$$= \frac{|\hat{E}_0|^2 \sin^2(\theta) \cdot 3\eta k^2 \cdot 4\pi r^2}{2\eta k^2 r^2 \cdot 4\pi |\hat{E}_0|^2}$$

$$= \frac{3}{2}\sin^2(\theta)$$

B. Radiation Resistance R

$$\langle P \rangle = \frac{|\hat{I}dl|^2 \eta k^2}{12\pi} = \frac{1}{2}|\hat{I}|^2 R$$

$$R = \frac{\eta}{6\pi}(kdl)^2 = \frac{\eta}{6\pi}4\pi^2\left(\frac{dl}{\lambda}\right)^2$$

$$= \frac{2}{3}\pi\eta\left(\frac{dl}{\lambda}\right)^2$$

In free space: $\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 120\pi \Rightarrow R = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2$

C. Effective Length of Antenna: $\hat{I}(z)$

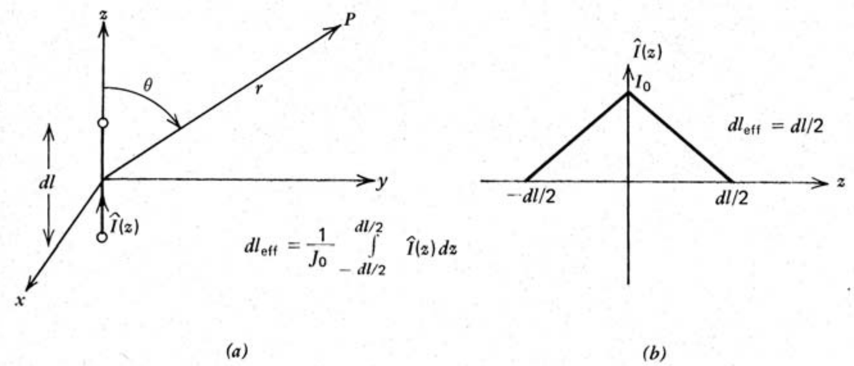

Figure 9-4 (*a*) If a point electric dipole has a nonuniform current distribution, the solutions are of the same form if we replace the actual dipole length dl by an effective length dl_{eff}. (*b*) For a triangular current distribution the effective length is half the true length.

$$\text{If } \frac{dl}{\lambda} \ll 1, \text{but current distribution depends on } z : \hat{I}(z)$$

$$\tilde{A}_z(r) = \int_{-\frac{dl}{2}}^{+\frac{dl}{2}} dz\, \frac{\mu \hat{I}(z) e^{-jkr_{QP}}}{4\pi r_{QP}} \approx \frac{\mu e^{-jkr}}{4\pi r} \int_{-\frac{dl}{2}}^{+\frac{dl}{2}} dz\, \hat{I}(z)$$

$$dl_{\text{eff}} = \frac{1}{\hat{I}(z=0)} \int_{-\frac{dl}{2}}^{+\frac{dl}{2}} dz\, \hat{I}(z)$$

Example:

$$\hat{I}(z) = \begin{cases} I_0\left(1 - \frac{2z}{dl}\right) & 0 < z < \frac{dl}{2} \\ I_0\left(1 + \frac{2z}{dl}\right) & -\frac{dl}{2} < z < 0 \end{cases}$$

$$\hat{I}(z=0) = I_0$$

$$dl_{\text{eff}} = \frac{1}{I_0} \int_{-\frac{dl}{2}}^{+\frac{dl}{2}} dz\, \hat{I}(z) = \frac{1}{\cancel{I_0}} \left(\frac{\cancel{I_0} dl}{2} \right) = \frac{dl}{2}$$

$$R = \frac{2\pi\eta}{3} \left(\frac{dl_{\text{eff}}}{\lambda} \right)^2 = 20\pi^2 \sqrt{\frac{\mu_r}{\epsilon_r}} \left(\frac{dl}{\lambda} \right)^2$$

D. Rayleigh Scattering (or, why is the sky blue?)

Electric field of light from the sun polarizes particles with dipole moment $\bar{p}$ in direction of $\bar{E}$.

$$\hat{p} = \hat{Q} dl = \frac{\hat{I} dl}{j\omega} \Rightarrow |\hat{I} dl| = |\omega \hat{p}|$$

$$\begin{aligned} \langle P \rangle &= \frac{|\hat{I} dl|^2 \eta k^2}{12\pi} \\ &= |\omega \hat{p}|^2 \frac{\eta \omega^2 \epsilon \mu}{12\pi} \\ &= \omega^4 |\hat{p}|^2 \frac{\eta \epsilon \mu}{12\pi} \end{aligned}$$

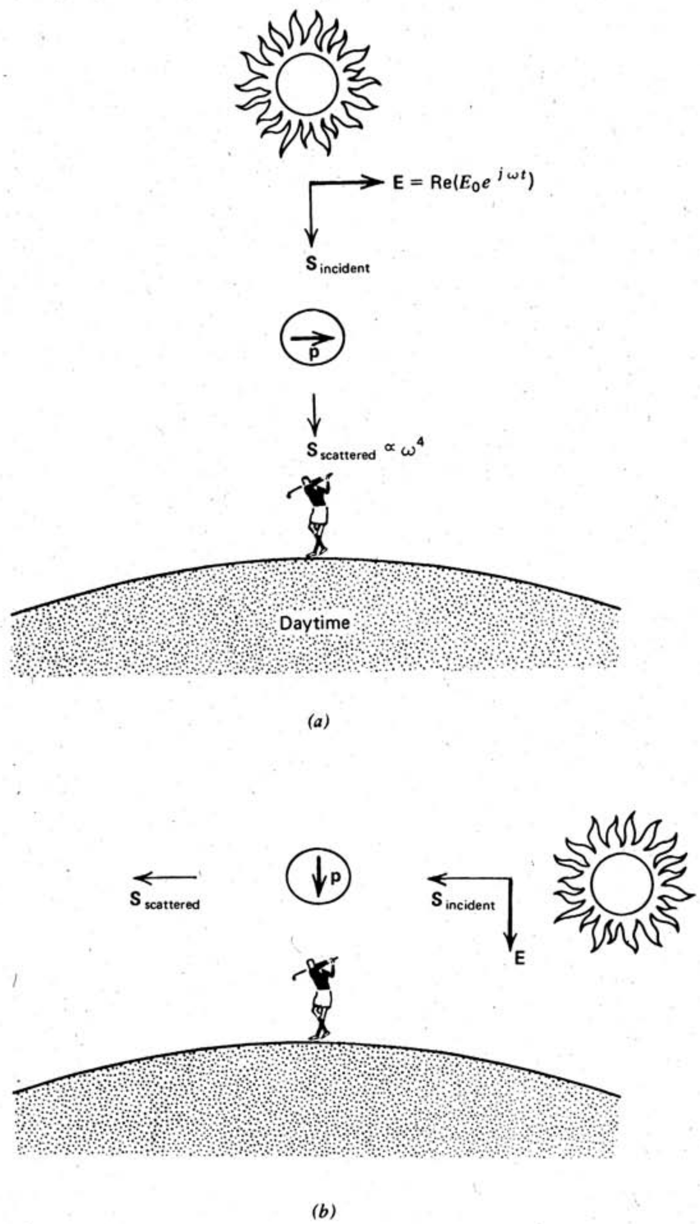

Figure 9-5 An incident electric field polarizes dipoles that then re-radiate their energy primarily perpendicular to the polarizing electric field. The time-average scattered power increases with the fourth power of frequency so shorter wavelengths of light are scattered more than longer wavelengths. (*a*) During the daytime an earth observer sees more of the blue scattered light so the sky looks blue (short wavelengths). (*b*) Near sunset the light reaching the observer lacks blue so the sky appears reddish (long wavelength).

Higher frequencies, and thus shorter wavelengths, are scattered in direction of S_{incident}:

$$\text{red:}\, \lambda \approx 633 \text{ nm}$$
$$\text{blue:}\, \lambda \approx 450 \text{ nm}$$

Blue light is thus scattered more than red.