Lecture 16 & 17: Electroquasistatic and Magnetoquasistatic Forces

I. EQS Energy Method of Forces

a) Circuit Point of View

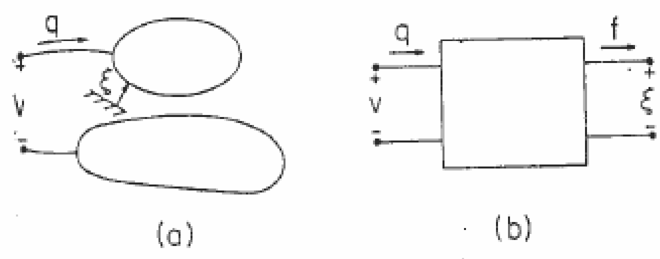


Figure 11.6.1 (a) Electroquasistatic system having one electrical terminal pair and one mechanical degree of freedom. (b) Schematic representation of EQS subsystem with coupling to external mechanical system represented by a mechanical terminal pair.

$$q = C(\xi)\,v$$

$$i = \frac{dq}{dt} = \frac{d}{dt}\left[C(\xi)\,v\right] = C(\xi)\frac{dv}{dt} + v\,\frac{dC(\xi)}{dt}$$

$$= C(\xi)\frac{dv}{dt} + v\,\frac{dC}{d\xi}\frac{d\xi}{dt}$$

$$P_{in} = vi = v\frac{d}{dt}\left[C(\xi)\,v\right] = C(\xi)\,v\frac{d\upsilon}{dt} + v^2\,\frac{dC}{d\xi}\frac{d\xi}{dt}$$

$$= C(\xi)\frac{d}{dt}\left(\frac{1}{2}v^2\right) + v^2\,\frac{dC}{d\xi}\frac{d\xi}{dt}$$

$$= \frac{d}{dt}\left[\frac{1}{2}C(\xi)\,v^2\right] + \frac{1}{2}v^2\,\frac{dC}{d\xi}\frac{d\xi}{dt}$$

$$= \underbrace{\frac{dW}{dt}}_{\substack{W=\text{energy}\\ \text{storage}}} + \underbrace{f_\xi\frac{d\xi}{dt}}_{\substack{\text{mechanical power}\\ (\text{force}\times\text{velocity})}}$$

$$W = \frac{1}{2} C(\xi) v^2 \,, \; f_\xi = \frac{1}{2} v^2 \frac{dC}{d\xi}$$

$$= \frac{1}{2} \frac{q^2}{C^2(\xi)} \frac{dC}{d\xi} = -\frac{1}{2} q^2 \frac{d}{d\xi}\left(\frac{1}{C(\xi)}\right)$$

b) Energy Point of View

$$vi = v \frac{dq}{dt} = \frac{dW_e}{dt} + f_\xi \frac{d\xi}{dt}$$

$$v dq = dW_e + f_\xi d\xi \Rightarrow dW_e = v dq - f_\xi d\xi$$

$$f_\xi = -\left.\frac{\partial W_e}{\partial \xi}\right|_{q=\text{constant}} \; ; \; v = \left.\frac{\partial W_e}{\partial q}\right|_{\xi=\text{constant}}$$

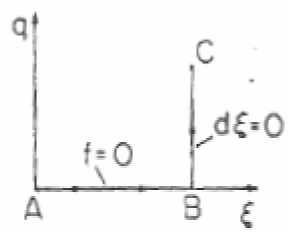


Figure 11.6.2 Path of line integration in state space (q, ξ) used to find energy at location C.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

$$W_e = -\int_{q=0} \overset{0}{\cancel{f_\xi}} d\xi + \int_{\xi=\text{constant}} v dq$$

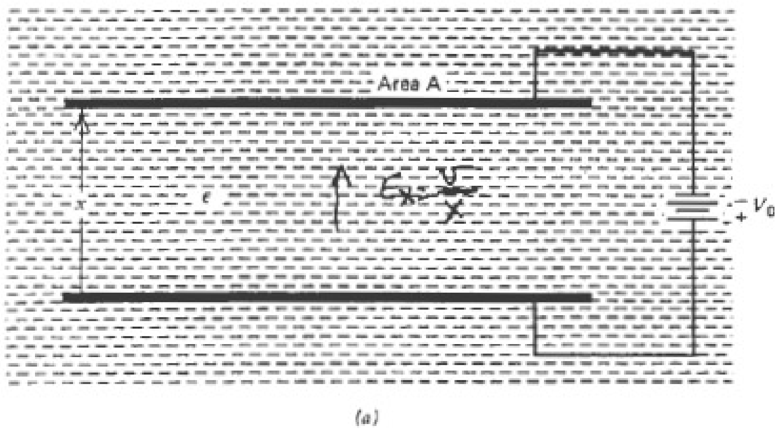
$$v = \frac{q}{C(\xi)}$$

$$W_e = \int_{\xi=\text{constant}} \frac{q}{C(\xi)} dq = \frac{1}{2} \frac{q^2}{C(\xi)}$$

$$f = -\left.\frac{\partial w_e}{\partial \xi}\right|_{q=\text{constant}} = -\frac{1}{2} q^2 \frac{d}{d\xi}\left(\frac{1}{C(\xi)}\right) = \frac{1}{2} \frac{q^2}{C^2(\xi)} \frac{dC(\xi)}{d\xi}$$

$$= \frac{1}{2} v^2 \frac{dC(\xi)}{d\xi}$$

II. Forces In Capacitors



$$\sigma_s = +\varepsilon E_x = \frac{+\varepsilon v}{x} \qquad \text{(Lower electrode)}$$

$$q = \sigma_s A = \varepsilon E_x A = \frac{\varepsilon v A}{x} = C(x)\,v$$

$$C(x) = \frac{\varepsilon A}{x}$$

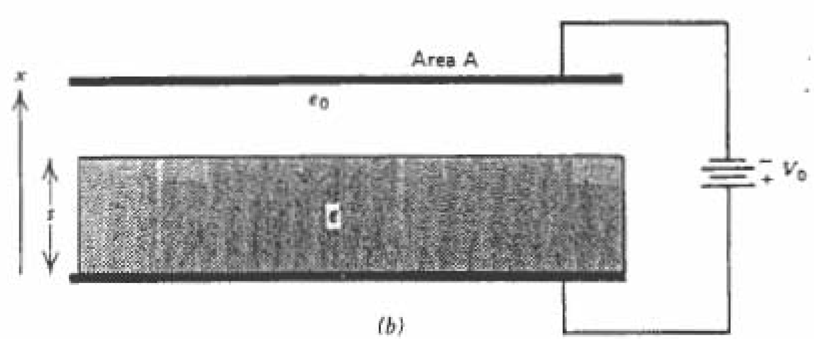


Figure 3-36 A parallel plate capacitor (*a*) immersed within a dielectric fluid or with (*b*) a free space region in series with a solid dielectric.

a) Coulombic force method on upper electrode:

$$f_x = \frac{1}{2}\sigma_s E_x A = -\frac{1}{2}\varepsilon E_x^2 A = -\frac{1}{2}\frac{\varepsilon v^2}{x^2}A$$

$\frac{1}{2}$ because E in electrode=0, E outside electrode = E_x

Take average

Energy method: $C(x) = \frac{\varepsilon A}{x}$

$$f_x = \frac{1}{2}v^2\frac{dC}{dx} = \frac{1}{2}v^2\varepsilon A\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{2}\frac{v^2\varepsilon A}{x^2}$$

$$v = \frac{q}{C(x)} = \frac{qx}{\varepsilon A} \Rightarrow f_x = -\frac{1}{2}\frac{\cancel{\varepsilon}\,\cancel{A}}{\cancel{x^2}}\frac{q^2\cancel{x^2}}{\varepsilon^{\cancel{2}}A^{\cancel{2}}} = -\frac{1}{2}\frac{q^2}{\varepsilon A}$$

b)

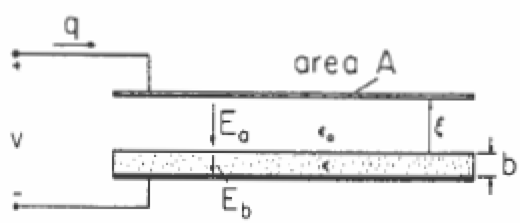


Figure 11.6.3 Specific example of EQS systems having one electrical and one mechanical terminal pair.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

$$\frac{1}{C(\xi)} = \frac{1}{C_a} + \frac{1}{C_b};\ C_a = \frac{\varepsilon_0 A}{\xi},\ C_b = \frac{\varepsilon A}{b}$$

$$= \frac{\xi}{\varepsilon_0 A} + \frac{b}{\varepsilon A}$$

$$= \frac{\varepsilon\,\xi + \varepsilon_0 b}{\varepsilon\,\varepsilon_0 A}$$

$$f_\xi = -\frac{1}{2}q^2\frac{d}{d\xi}\left(\frac{1}{C(\xi)}\right) = \frac{-\frac{1}{2}q^2}{\varepsilon\,\varepsilon_0 A}\frac{d}{d\xi}\left(\varepsilon\,\xi + \varepsilon_0 b\right) = -\frac{1}{2}\frac{q^2}{\varepsilon_0 A}$$

$$f_\xi = \frac{1}{2}v^2\frac{d}{d\xi}\left(C(\xi)\right) = \frac{1}{2}v^2\frac{d}{d\xi}\left[\frac{\varepsilon\,\varepsilon_0 A}{\varepsilon\,\xi + \varepsilon_0 b}\right] = -\frac{1}{2}\frac{v^2\varepsilon^2\varepsilon_0 A}{\left(\varepsilon\,\xi + \varepsilon_0 b\right)^2}$$

III. Energy Conversion Cycles

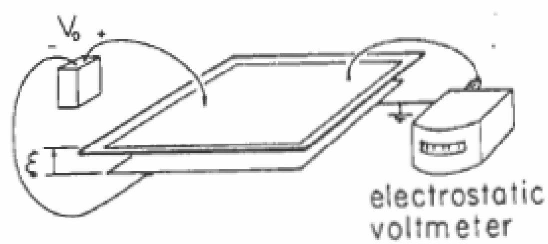


Figure 11.6.4 Apparatus used to demonstrate amplification of voltage as the upper electrode is raised. (The electrodes are initially charged and then the voltage source is removed so q = constant.) The electrodes, consisting of foil mounted on insulating sheets, are about 1 m × 1 m, with the upper one insulated from the frame, which is used to control its position. The voltage is measured by the electrostatic voltmeter, which "loads" the system with a capacitance that is small compared to that of the electrodes and (at least on a dry day) a negligible resistance.

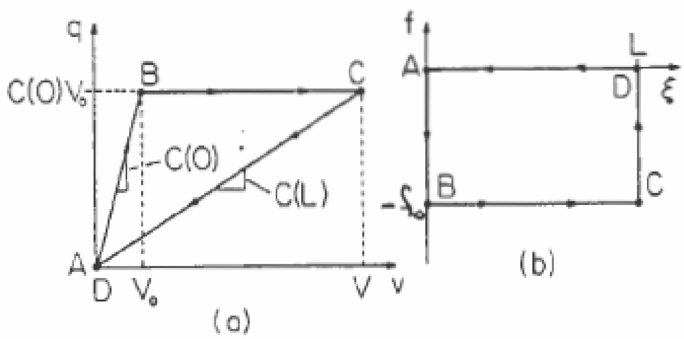


Figure 11.6.5 Closed paths followed in cyclic conversion of energy from mechanical to electrical form: (a) in (q, v) plane; and (b) in (f, ξ) plane.

- $A \rightarrow B$. With $v = 0$, the upper electrode rests on the plastic sheet. A voltage V_o is applied.
- $B \rightarrow C$. With the voltage source removed so that the upper electrode is electrically isolated, it is raised to the position $\xi = L$.
- $C \rightarrow D$. The upper electrode is shorted, so that its voltage returns to zero.
- $D \rightarrow A$. The upper electrode is returned to its original position at $\xi = 0$.

Is electrical energy converted to mechanical form, or vice versa?

$$\oint v\,dq = \oint dW_e^{\nearrow 0} + \oint f_\xi d\xi$$

$$\oint v\,dq = \oint f_\xi d\xi$$

$$\underbrace{\oint v\,dq,\ \oint f_\xi d\xi > 0}$$

Electric energy in, mechanical energy out.

$$\underbrace{\oint v\,dq,\ \oint f_\xi d\xi < 0}$$

Electric power out, mechanical energy in.

$$\oint v\,dq = \int_A^B v\,dq + \int_C^D v\,dq = \frac{1}{2}C(0)V_0^2 - \frac{1}{2}C(L)V^2$$

$$C(0)V_0 = C(L)V$$

$$\oint v\,dq = \frac{1}{2}C(0)V_0^2\left[1 - \frac{\cancel{C(L)}C(0)}{C^{\cancel{2}}(L)}\right] = \frac{1}{2}C(0)V_0^2\left[1 - \frac{C(0)}{C(L)}\right]$$

$$\frac{C(0)}{C(L)} = \frac{\varepsilon \cancel{A}\left(L + b\dfrac{\varepsilon_0}{\varepsilon}\right)}{b\left(\varepsilon_0 \cancel{A}\right)}$$

$$\oint v\,dq = \frac{1}{2}C(0)V_0^2\left[1 - \frac{\left(L + b\dfrac{\varepsilon_0}{\varepsilon}\right)\varepsilon}{\varepsilon_0 b}\right] = -\frac{1}{2}C(0)V_0^2\,\frac{\varepsilon}{\varepsilon_0}\frac{L}{b} < 0 \ \text{(electric energy out)}$$

$$\oint f d\xi = -f_0 L$$

$$f_0 = +\frac{1}{2}\frac{q^2}{\varepsilon_0 A} = +\frac{1}{2}\frac{C^2(0)V_0^2}{\varepsilon_0 A} = +\frac{1}{2}C(0)V_0^2\left[\frac{\varepsilon \cancel{A}}{b\varepsilon_0 \cancel{A}}\right]$$

$$\oint f d\xi = -\frac{1}{2}C(0)V_0^2\,\frac{\varepsilon L}{\varepsilon_0 b} = \oint v\,dq$$

$\oint f d\xi < 0 \Rightarrow$ mechanical energy out is negative means mechanical energy is put in

Mechanical energy is converted to electrical energy

IV. Force on a Dielectric Material

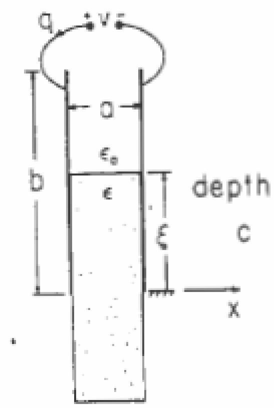


Figure 11.6.6 Slab of dielectric partially extending between capacitor plates. The spacing, a, is much less than either b or the depth c of the system into the paper. Further, the upper surface at ξ is many spacings a away from the upper and lower edges of the capacitor plates, as is the lower surface as well.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

$$C(\xi) = \frac{\varepsilon_0 (b - \xi) c}{a} + \frac{\varepsilon \xi c}{a}$$

$$f_\xi = \frac{1}{2} v^2 \frac{dC(\xi)}{d\xi}$$

$$= \frac{1}{2} v^2 \frac{c}{a} (\varepsilon - \varepsilon_0)$$

In equilibrium:

$$f_\xi = \frac{1}{2} v^2 \frac{c}{a} (\varepsilon - \varepsilon_0) = \underbrace{\rho g \xi a c}_{\text{fluid weight}}$$

(Mass density: ρ)

$$\xi = \frac{1}{2} \frac{v^2 (\varepsilon - \varepsilon_0)}{\rho g a^2}$$

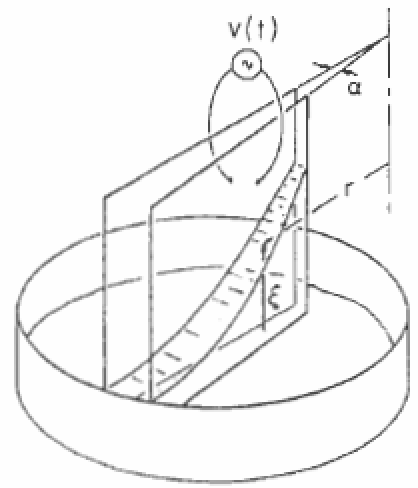


Figure 11.6.7 In a demonstration of the polarization force, a pair of conducting transparent electrodes are dipped into a liquid (corn oil dyed with food coloring). They are closer together at the upper right than at the lower left, so when a voltage is applied, the electric field intensity decreases with increasing distance, r, from the apex. As a result, the liquid is seen to rise to a height that varies as $1/r^2$. The electrodes are about 10 cm × 10 cm, with an electric field exceeding the nominal breakdown strength of air at atmospheric pressure, 3×10^6 V/m. The experiment is therefore carried out under pressurized nitrogen.

$a \rightarrow \alpha r$

$$\xi = \frac{1}{2}\frac{v^2(\varepsilon - \varepsilon_0)}{\rho g \alpha^2 r^2}$$

V. Physical Model of Forces on Dielectrics

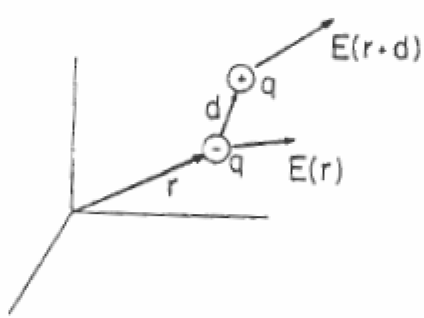


Figure 11.8.1 An electric dipole experiences a net electric force if the positive charge q is subject to an electric field $\mathbf{E}(\mathbf{r}+\mathbf{d})$ that differs from $\mathbf{E}(\mathbf{r})$ acting on the negative charge q.

$$\bar{f}_{dipole} = q\left[\bar{E}(\bar{r}+\bar{d}) - \bar{E}(\bar{r})\right]$$

$$= q\left[\bar{E}(\bar{r}) + \bar{d} \cdot \nabla \bar{E}(\bar{r}) - \bar{E}(\bar{r})\right]$$

$$= q\left(\bar{\ } \cdot \nabla\right)\bar{E}$$

$$= \left(\bar{p} \cdot \nabla\right)\bar{E} \qquad \text{Kelvin force}$$

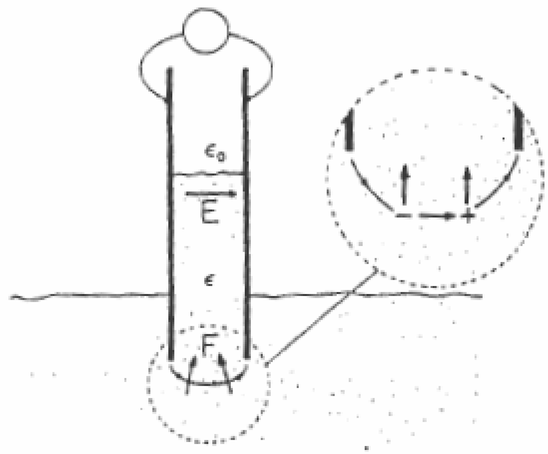
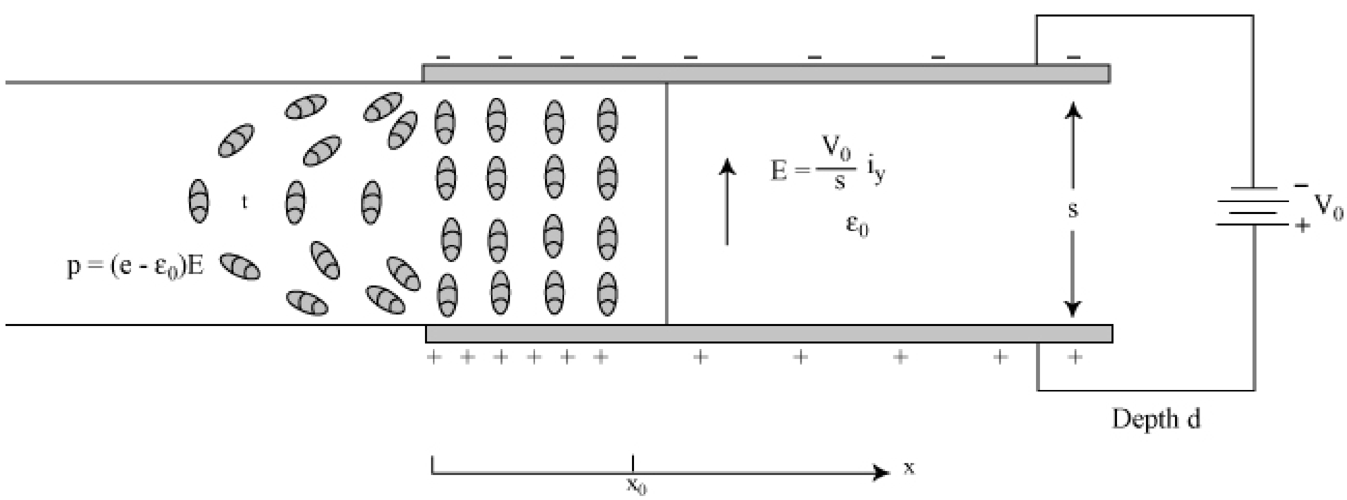
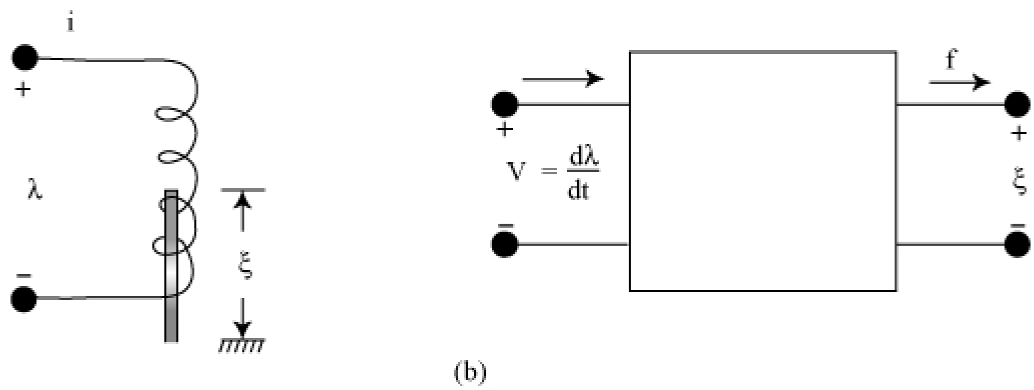


Figure 11.9.4 In terms of the Kelvin force density, the dielectric liquid is pushed into the field region between capacitor plates because of the forces on individual dipoles in the fringing field.



A linear dielectric is always attracted into a free space capacitor because of the net force on dipoles in the nonuniform field. The dipoles are now aligned with the electric field. no matter the voltage polarity.

VI. MQS Energy Method of Forces



(a) Magnetoquasistatic system having one electrical terminal pair and one mechanical degree of freedom. (b) Schematic representation of MQS subsystem with coupling to external mechanical system represent by a mechanical terminal pair.

A. Circuit Approach

$$v = \frac{d\lambda}{dt} = \frac{d}{dt}\left[L(\xi)i\right] = L(\xi)\frac{di}{dt} + i\frac{dL(\xi)}{dt}$$

$$p = vi = L(\xi)i\frac{di}{dt} + i^2\frac{dL(\xi)}{dt}$$

$$= L(\xi)\frac{d}{dt}\left(\frac{1}{2}i^2\right) + i^2\frac{dL(\xi)}{dt}$$

$$= \frac{d}{dt}\left[\frac{1}{2}L(\xi)i^2\right] + \frac{1}{2}i^2\frac{dL(\xi)}{dt}$$

$$= \frac{d}{dt}\left[\frac{1}{2}L(\xi)i^2\right] + \frac{1}{2}i^2\frac{dL(\xi)}{d\xi}\frac{d\xi}{dt}$$

$$vi = \frac{dW_m}{dt} + f_\xi\frac{d\xi}{dt} \Rightarrow W_m = \frac{1}{2}L(\xi)i^2, \quad f_\xi = \frac{1}{2}i^2\frac{dL(\xi)}{d\xi}$$

$$\lambda = L(\xi)i \Rightarrow f_\xi = \frac{1}{2}i^2\frac{dL(\xi)}{d\xi}$$

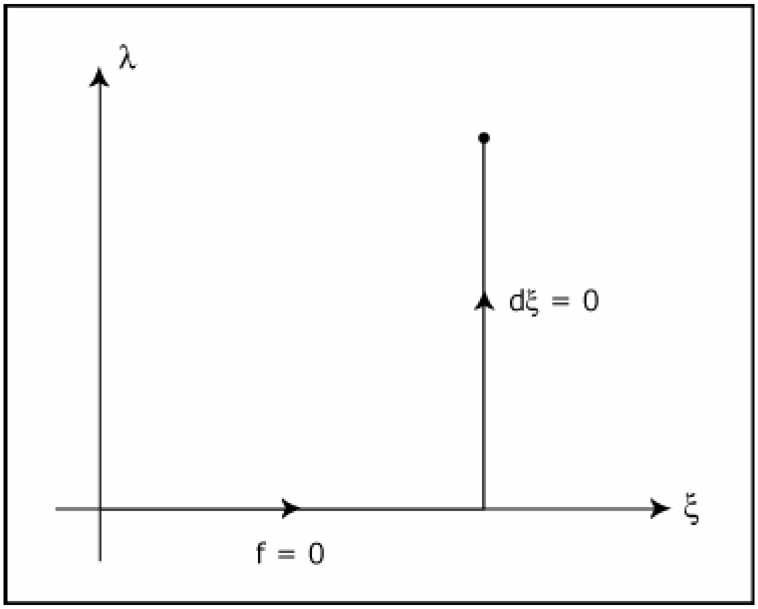
$$= \frac{1}{2}\frac{\lambda^2}{L^2(\xi)}\frac{dL(\xi)}{d\xi}$$

$$= -\frac{1}{2}\lambda^2\frac{d}{d\xi}\left[1/L(\xi)\right]$$

B. Energy Method

$$vi = i\frac{d\lambda}{dt} = \frac{dW_m}{dt} + f_\xi \frac{d\xi}{dt} \Rightarrow dW_m = i\,d\lambda - f_\xi\, d\xi$$

$$f_\xi = -\left.\frac{\partial W_m}{\partial \xi}\right|_{\lambda = \text{constant}} \quad , \quad i = \left.\frac{\partial W_m}{\partial \lambda}\right|_{\xi=\text{constant}}$$



$$W_m = \int_{\lambda=0} -\overset{\nearrow 0}{f_\xi}\, d\xi + \int_{\xi=\text{constant}} i\ d\lambda$$

$$i = \frac{\lambda}{L(\xi)}$$

$$W_m = \int_{\xi=\text{constant}} \frac{\lambda}{L(\xi)}\ d\lambda = \frac{\lambda^2}{2L(\xi)}$$

$$f_\xi = \left.\frac{-\partial W_m}{\partial \xi}\right|_{\lambda=\text{constant}} = -\frac{1}{2}\lambda^2 \frac{d}{d\xi}\left(\frac{1}{L(\xi)}\right)$$

$$= -\frac{1}{2}\lambda^2 \left(-\frac{1}{L^2(\xi)}\right)\frac{dL(\xi)}{d\xi}$$

$$= \frac{1}{2} i^2 \frac{dL(\xi)}{d\xi}$$

VII. Force on a Wire over a Perfectly Conducting Plane

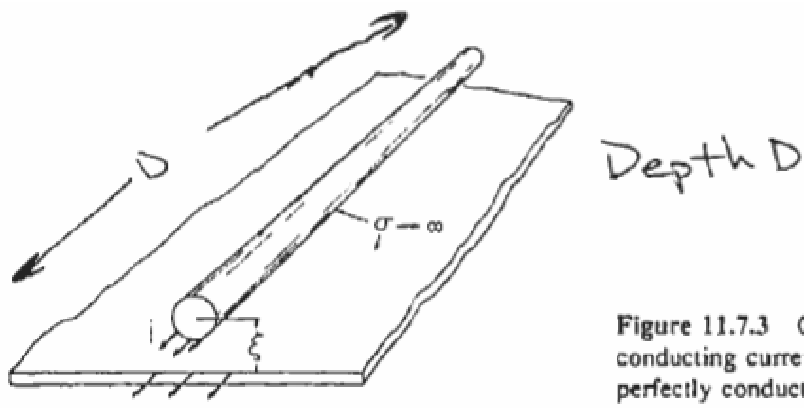


Figure 11.7.3 Cross-section of perfectly conducting current-carrying wire over a perfectly conducting ground plane.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

$$L(\xi) = \frac{\mu_0 D}{2\pi} \ln\left[\frac{\xi}{R} + \sqrt{\left(\frac{\xi}{R}\right)^2 - 1}\right]$$

[See Haus & Melcher p. 343, take ½ of Eq. (12) which is the inductance between 2 cylinders]

A. Energy Method

$$f_\xi = \frac{1}{2} i^2 \frac{dL(\xi)}{d\xi} = \frac{\mu_0 i^2 D}{4\pi R} \frac{1}{\sqrt{\left(\xi/R\right)^2 - 1}}$$

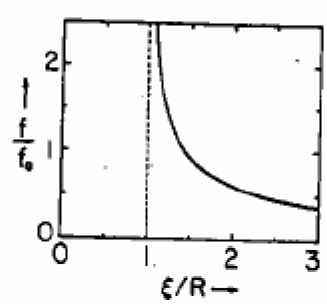
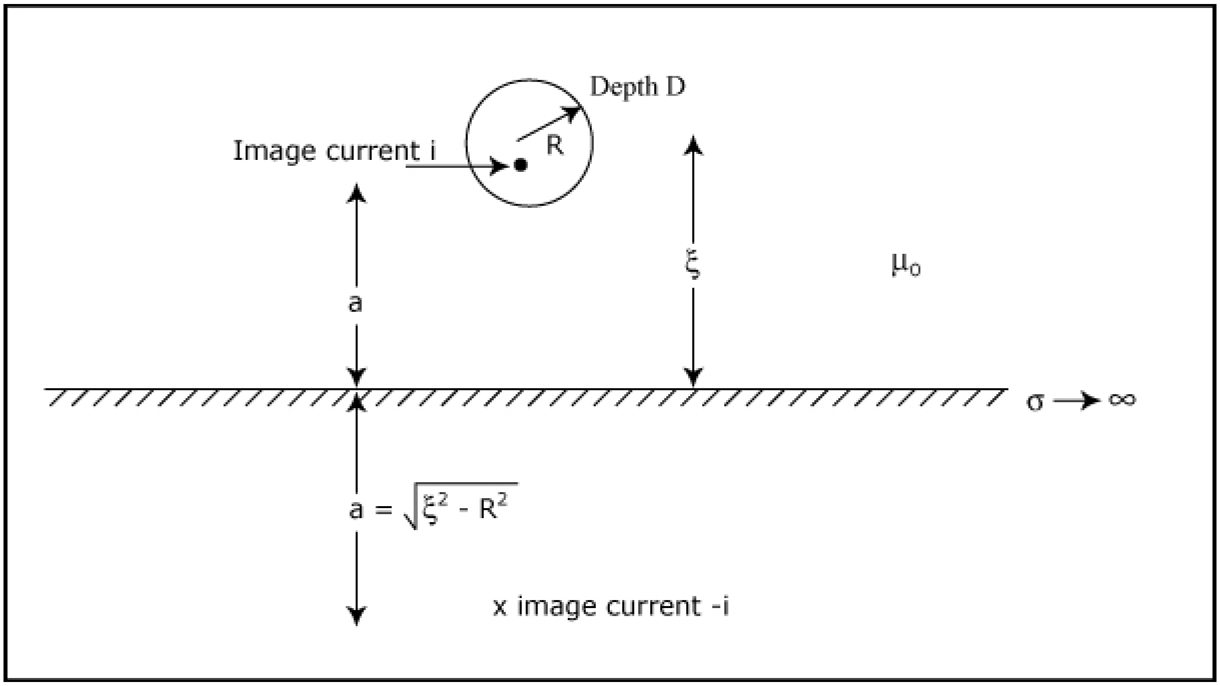


Figure 11.7.4 The force tending to levitate the wire of Figure 11.7.3 as a function of the distance to the ground plane normalized to the radius R of wire.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

B. Method of Images Approach with Lorentz Force



$$f_\xi = iD\left(\frac{\mu_0 i}{2\pi(2a)}\right) = \frac{\mu_0 i^2 D}{4\pi a} = \frac{\mu_0 i^2 D}{4\pi\sqrt{\xi^2 - R^2}}$$

C. Demonstration: Steady State Magnetic Leviation

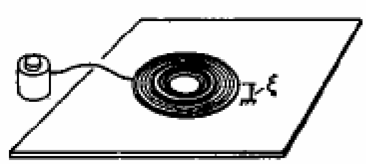


Figure 11.7.5 When the pancake coil is driven by an ac current, it floats above the aluminum plate. In this experiment, the coil consists of 250 turns of No. 10 aluminum wire with an outer radius of 16 cm and an inner one of 2.5 cm. The aluminum sheet has a thickness of 1.3 cm. With a 60 Hz current *i* of about 20 amp rms, the height above the plate is 2 cm.

VIII. One Turn Loop

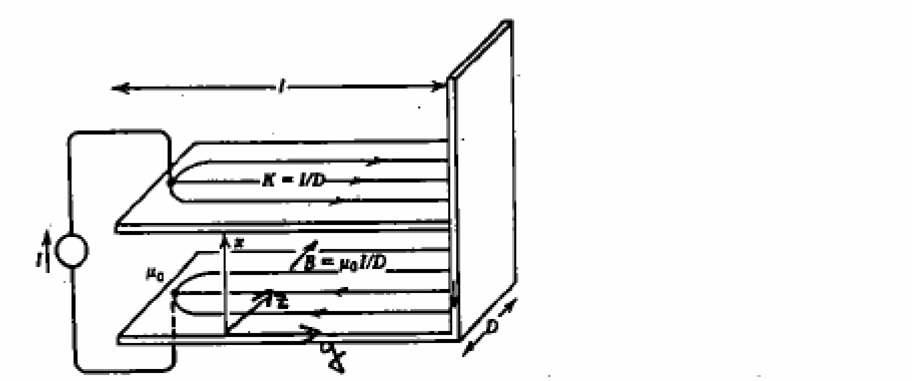


Figure 6-35 The magnetic force on a current-carrying loop tends to expand the loop.

From *Electromagnetic Field Theory: A Problem Solving Approach*,
by Markus Zahn, 1987. Used with permission.

$$H_z = \frac{I}{D}\ , \quad \Phi = \mu_0 H_z x l\ , \quad L(x) = \Phi/I = \frac{\mu_0 x l}{D}$$

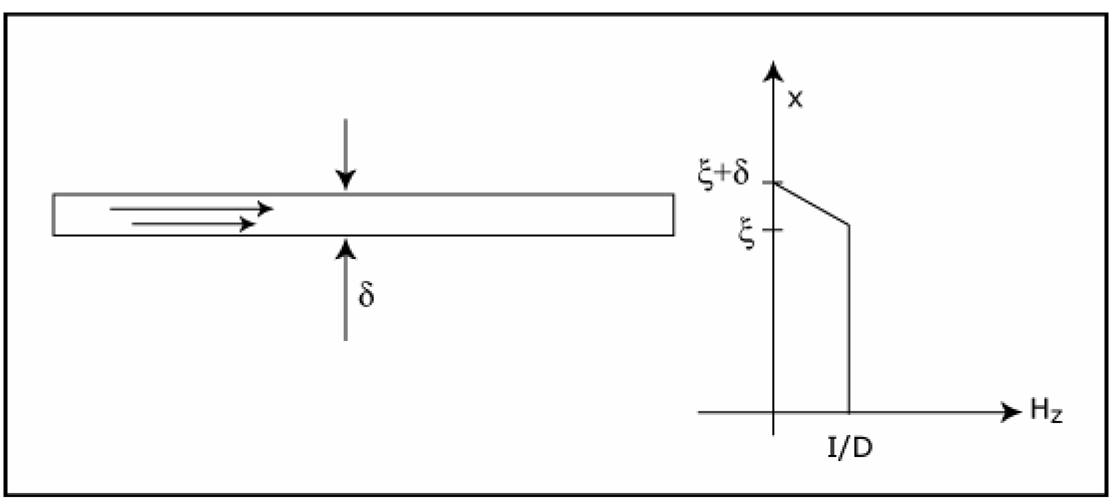
$$= \frac{\mu_0 x l}{D} I$$

A. Energy Method

$$f_x = \frac{1}{2} I^2 \frac{dL(x)}{dx} = \frac{1}{2} I^2 \frac{\mu_0 l}{D}$$

B. Lorentz Force Law

$$\bar{f} = \int_V \bar{J} \times \bar{B}\ dV$$



Model surface current $K_y = \frac{I}{D}$ as volume current of small thickness δ

$$J_y = \frac{I}{D\delta}$$

$$\nabla \times \overline{H} = \overline{J} \Rightarrow \frac{\partial H_z}{\partial x} = -J_y = -\frac{I}{D\delta} \Rightarrow H_z = -\frac{I}{D\delta}\left(x - (\xi + \delta)\right)$$

$$f_x = \int_V J_y \, \mu_0 \, H_z \, dx \, dy \, dx$$

$$= \int_{x=\xi}^{\xi+\delta} \frac{I}{D\delta}\left(\frac{-\mu_0 I}{\not{D}\delta}\right)\left(x - (\xi + \delta)\right) I \not{D} \, dx$$

$$= \frac{-\mu_0 I^2 l}{D\delta^2}\left[\frac{x^2}{2} - (\xi + \delta)\, x\right]\Bigg|_{x=\xi}^{\xi+\delta}$$

$$= \frac{-\mu_0 I^2 l}{D\delta^2}\left[\frac{(\xi + \delta)^2}{2} - \frac{\xi^2}{2} - (\xi + \delta)^2 + \xi(\xi + \delta)\right]$$

$$= \frac{-\mu_0 I^2 l}{D\delta^2}\left[-\frac{1}{2}(\xi + \delta)^2 + \frac{\xi^2}{2} + \xi\delta\right]$$

$$= \frac{-\mu_0 I^2 l}{D\delta^2}\left[-\frac{1}{2}\delta^2\right]$$

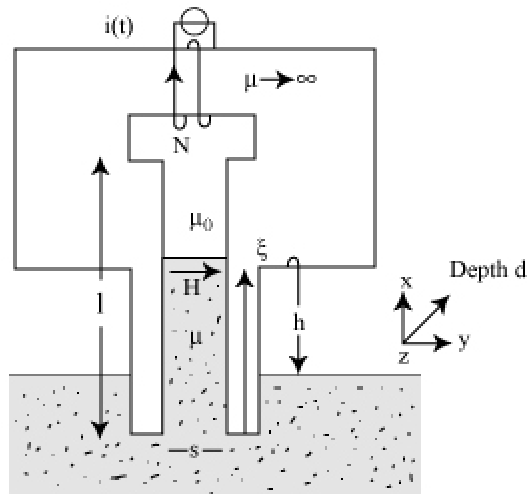
$$= +\frac{1}{2}\frac{\mu_0 I^2 l}{D} \Rightarrow \overline{f} = \int_S \frac{1}{2}\overline{K} \times \overline{B} \, dS$$

½ comes from integrating uniform volume current over small thickness δ

General formula: $\overline{f} = \int_S \overline{K} \times \overline{B}_{av} \, dS$

For our case: $B_{av} = \frac{\cancelto{0}{B_{metal}} + B_{air}}{2} = \frac{1}{2} B_{air}$

IX. Lifting of Magnetic Fluid



In an experiment this is the magnetic analog of that shown in figure 11.94 a magnetizable liquid is pushed upward into the field region between the pole faces by the forces on magnetic dipoles in the fringing region at the bottom.

A. Energy Method Approach

$$H = \frac{Ni}{S}$$

$$\Phi = H\left[\mu\xi + \mu_0 \left(l - \xi\right)\right] d$$

$$= \frac{Nd}{s}\left[\mu\xi + \mu_0 \left(l - \xi\right)\right] i$$

$$\lambda = N\Phi = \frac{N^2 d}{s}\left[\mu\xi + \mu_0 \left(l - \xi\right)\right] i$$

$$L\left(\xi\right) = \frac{\lambda}{i} = \frac{N^2 d}{s}\left[\mu\xi + \mu_0 \left(l - \xi\right)\right]$$

$$f_\xi = \frac{1}{2} i^2 \frac{dL}{d\xi} = \frac{1}{2}\frac{N^2 i^2 d}{s}\left(\mu - \mu_0\right)$$

$$f_\xi = \rho_m\, g\, h\, d\, s = \frac{1}{2}\frac{N^2 i^2 d}{s}\left(\mu - \mu_0\right)$$

$$h = \frac{1}{2}\,\frac{N^2 i^2 \cancel{d}}{s^2 \rho_m\, g\, \cancel{d}}\left(\mu - \mu_0\right)$$

B. Magnetization force

$$F_x = \mu_0 \left(\overline{M} \bullet \nabla\right) H_x$$

$$= \mu_0 \left[M_x \frac{\partial H_x}{\partial x} + M_y \frac{\partial H_x}{\partial y} + \underbrace{M_z \frac{\partial H_x}{\partial z}}_{\frac{\partial}{\partial z} = 0} \right]$$

$$\nabla \times \overline{H} = \overline{J} = 0 \Rightarrow \frac{\partial H_x}{\partial y} = \frac{\partial H_y}{\partial x}$$

$$F_x = \mu_0 \left[M_x \frac{\partial H_x}{\partial x} + M_y \frac{\partial H_y}{\partial x} \right]$$

$$\overline{B} = \mu \overline{H} = \mu_0 \left(\overline{H} + \overline{M}\right) \Rightarrow \overline{M} = \left(\frac{\mu}{\mu_0} - 1\right)\overline{H}$$

$$F_x = \mu_0 \left[\left(\frac{\mu}{\mu_0} - 1\right) H_x \frac{\partial H_x}{\partial x} + \left(\frac{\mu}{\mu_0} - 1\right) H_y \frac{\partial H_y}{\partial x} \right]$$

$$= \mu_0 \left(\frac{\mu}{\mu_0} - 1\right) \frac{\partial}{\partial x}\left[\frac{1}{2}\left(H_x^2 + H_y^2\right)\right]$$

$$f_x = \int F_x \, dx\, dy\, dz$$

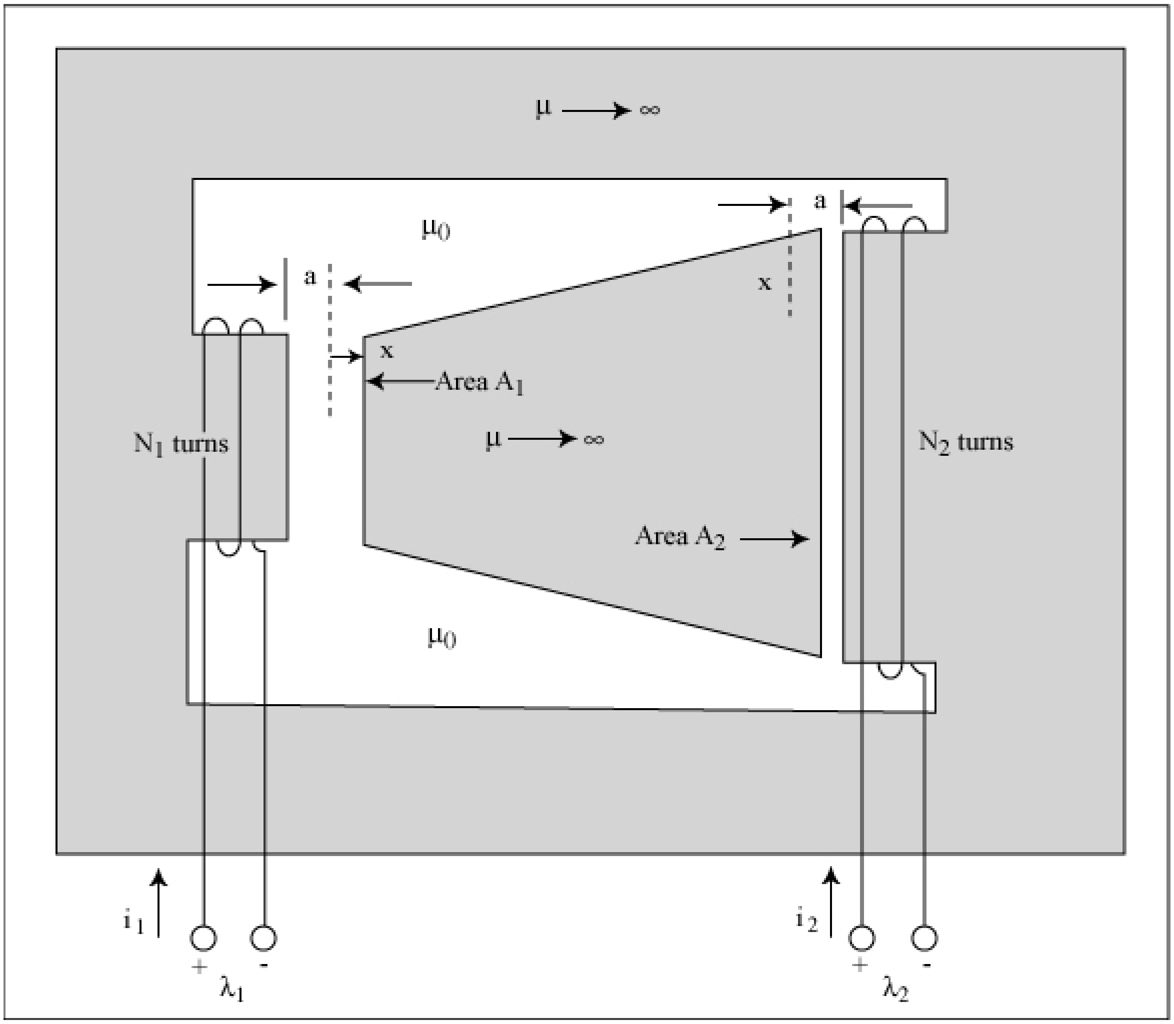
$$= \frac{\left(\mu - \mu_0\right)}{2} \int_{x=-\infty}^{h} \int_{y=0}^{s} \int_{z=0}^{d} \frac{\partial}{\partial x}\left(H_x^2 + H_y^2\right) dx\, dy\, dz$$

$$= \frac{\left(\mu - \mu_0\right)}{2} d s \left(H_x^2 + H_y^2\right)\Big|_{x=-\infty}^{h}$$

$$= \frac{\left(\mu - \mu_0\right)}{2} d \not{s} \frac{N^2 i^2}{s^{\not{2}}}$$

$$= \frac{1}{2}\left(\mu - \mu_0\right) d \frac{N^2 i^2}{s}$$

X. Magnetic Actuator



$$\oint_C \overline{H} \cdot d\overline{s} = H_1\left(x + a\right) + H_2\left(a - x\right) = N_1 i_1 + N_2 i_2$$

$$\mu_0 H_1 A_1 = \mu_0 H_2 A_2 \Rightarrow H_1 = \frac{H_2 A_2}{A_1}$$

$$H_2\left[\left(a - x\right) + \left(a + x\right)\frac{A_2}{A_1}\right] = N_1 i_1 + N_2 i_2$$

$$H_2 = \frac{\left(N_1 i_1 + N_2 i_2\right) A_1}{A_1\left(a - x\right) + \left(a + x\right) A_2}$$

$$H_1 = \frac{(N_1 i_1 + N_2 i_2) A_2}{A_1 (a-x) + (a+x) A_2}$$

$$\lambda_1 = N_1 \mu_0 H_1 A_1 = \frac{\mu_0 N_1 A_1 A_2 (N_1 i_1 + N_2 i_2)}{A_1 (a-x) + (a+x) A_2}$$

$$\lambda_2 = N_2 \mu_0 H_2 A_2 = \frac{\mu_0 N_2 A_1 A_2 (N_1 i_1 + N_2 i_2)}{A_1 (a-x) + (a+x) A_2}$$

$$\lambda_1 = L_1(x) i_1 + M(x) i_2$$

$$\lambda_2 = M(x) i_1 + L_2(x) i_2$$

$$L_1(x) = \frac{\mu_0 A_1 A_2 N_1^2}{A_1 (a-x) + (a+x) A_2} \; ; \quad L_2(x) = \frac{\mu_0 A_1 A_2 N_2^2}{A_1 (a-x) + (a+x) A_2} \; ; \quad M(x) = \frac{\mu_0 A_1 A_2 N_1 N_2}{A_1 (a-x) + (a+x) A_2}$$

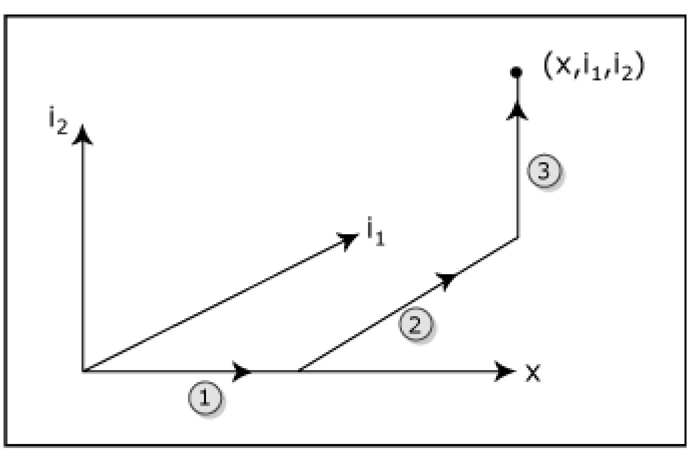
$$= \sqrt{L_1(x) L_2(x)}$$

$$dw = i_1 \, d\lambda_1 + i_2 \, d\lambda_2 - f \, dx$$

$$d\underbrace{(i_1 \lambda_1 + i_2 \lambda_2 - w)}_{w' \text{ (co–energy)}} = \lambda_1 \, di_1 + \lambda_2 \, di_2 + f \, dx$$

$$dw' = \lambda_1 \, di_1 + \lambda_2 \, di_2 + f \, dx$$

$$f = + \left. \frac{\partial w'}{\partial x} \right|_{i_1, i_2 \text{ constant}}$$



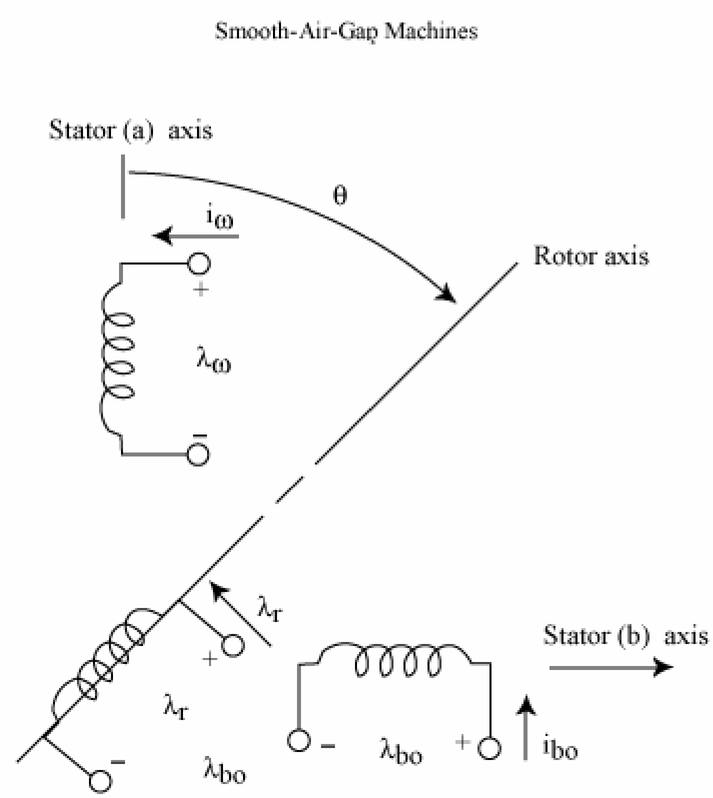
$$dw' = \int\limits_{\substack{1 \\ i_1 = i_2 = 0}} f\,dx + \int\limits_{\substack{2 \\ i_2 = 0 \\ x = \text{constant}}} \lambda_1\,di_1 + \int\limits_{\substack{3 \\ i_1 = \text{constant} \\ x = \text{constant}}} \lambda_2\,di_2$$

$$dw' = \int\limits_{\substack{i_2 = 0 \\ x = \text{constant}}} L_1(x) i_1\,di_1 + \int\limits_{\substack{i_1 = \text{constant} \\ x = \text{constant}}} \left(M(x) i_1 + L_2(x) i_2\right) di_2$$

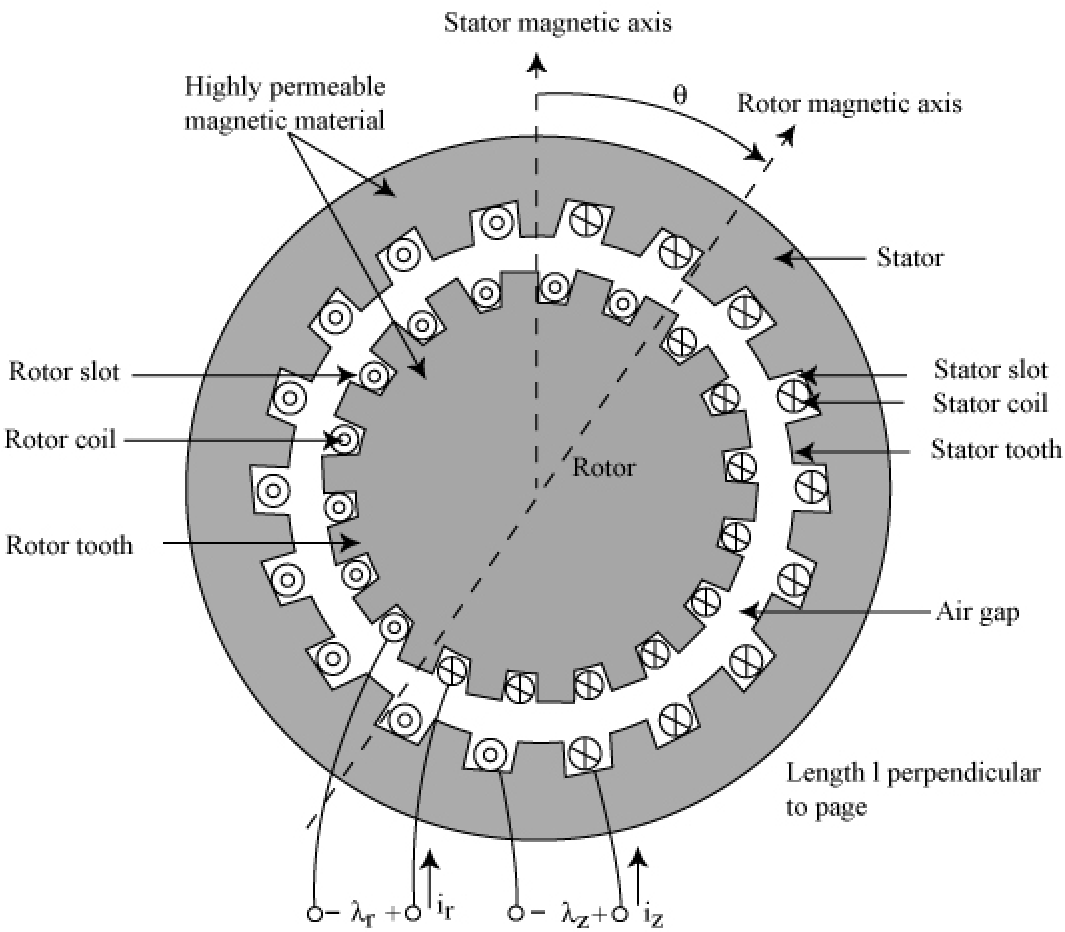
$$= \frac{1}{2} L_1(x) i_1^2 + M(x) i_1 i_2 + \frac{1}{2} L_2(x) i_2^2$$

$$f = + \left.\frac{\partial w'}{\partial x}\right|_{i_1, i_2} = \frac{1}{2} i_1^2 \frac{dL_1}{dx} + \frac{1}{2} i_2^2 \frac{dL_2}{dx} + i_1 i_2 \frac{dM}{dx}$$

XI. Synchronous Machine



Schematic representation of smooth-air-gap synchronous machine with field (dc) winding on the rotor and balanced two phase stator (armature) winding.



Geometry of smooth-air-gap rotating machine showing distributed windings on stator and rotor of a single-phase machine.

$$\lambda_{as} = L_s\, i_{as} + M i_r \cos\theta$$

$$\lambda_{bs} = L_s\, i_{bs} + M i_r \sin\theta$$

$$\lambda_r = L_r\, i_r + M\left(i_{as}\sin\theta + i_{bs}\sin\theta\right)$$

$$dw = i_{as}\, d\lambda_{as} + i_{bs}\, d\lambda_{bs} + i_r\, d\lambda_r - T^e\, d\theta$$

$$d\left(w - i_{as}\,\lambda_{as} - i_{bs}\,\lambda_{bs} - i_r\,\lambda_r\right) = -dw'$$

$$w' = i_{as}\,\lambda_{as} + i_{bs}\,\lambda_{bs} + i_r\,\lambda_r - w \qquad \text{co–energy}$$

$$dw' = \lambda_{as}\, di_{as} + \lambda_{bs}\, di_{bs} + \lambda_r\, di_r + T^e\, d\theta$$

$$w' = \underset{\substack{i_{as}=0\\ i_{bs}=0\\ i_r=0}}{\int} T\,d\theta + \underset{\substack{\theta=\text{constant}\\ i_{bs}=0\\ i_r=0}}{\int} \lambda_{as}\,di_{as} + \underset{\substack{\theta=\text{constant}\\ i_{as}=\text{constant}\\ i_r=0}}{\int} \lambda_{bs}\,di_{bs} + \underset{\substack{\theta=\text{constant}\\ i_{as}=\text{constant}\\ i_{bs}=\text{constant}}}{\int} \lambda_r\,di_r$$

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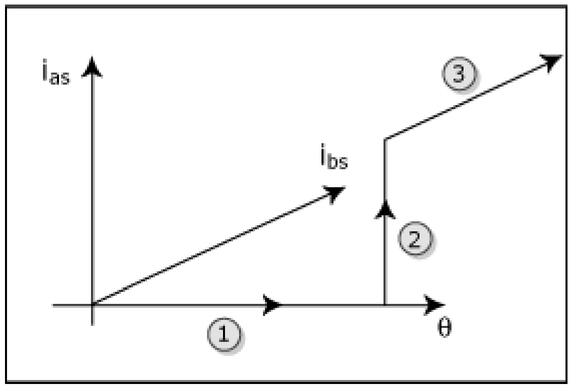


$$w' = \underset{\substack{i_{as}=0\\ i_{bs}=0\\ i_r=0}}{\int} \overset{0}{\cancel{T}}\,d\theta + \int L_s\,i_{as}\,di_{as} + \int L_s\,i_{bs}\,di_{bs} + \underset{\substack{\theta=\text{constant}\\ i_{as}=\text{constant}\\ i_{bs}=\text{constant}}}{\int} \left[L_r i_r + M\left(i_{as}\cos\theta + i_{bs}\sin\theta\right)\right] di_r$$

① ② ③ ④

$$w' = \frac{1}{2}L_s\,i_{as}^{\;2} + \frac{1}{2}L_s\,i_{bs}^{\;2} + \frac{1}{2}L_r\,i_r^{\,2} + M i_r\left(i_{as}\cos\theta + i_{bs}\sin\theta\right)$$

$$T^e = +\left.\frac{\partial w'}{\partial\theta}\right|_{i_{as},\,i_{bs},\,i_r} = M i_r\left(-i_{as}\sin\theta + i_{bs}\cos\theta\right)$$

Balanced 2 phase currents

$$i_{as} = I_s\cos\omega t\,,\;\; i_{bs} = I_s\sin\omega t\,,\;\; i_r = I_r\,,\;\; \theta = \omega_m t + \gamma$$

$$T^e = M I_r\,I_s\left(-\cos\omega t\;\sin\theta + \sin\omega t\;\cos\theta\right) = M I_r\,I_s\sin\left(\omega t - \theta\right)$$
$$= M I_r\,I_s\sin\left(\left(\omega - \omega_m\right)t - \gamma\right)$$

$$< T^e > \neq 0 \Rightarrow \omega = \omega_m$$

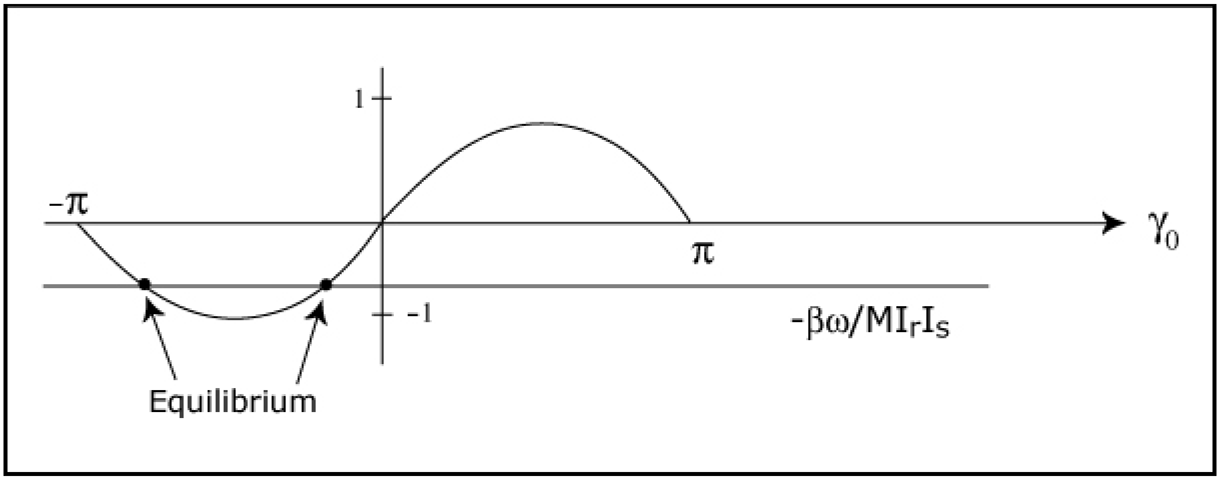
$$T^e = -M I_r\,I_s\sin\gamma$$

$$J\frac{d^2\theta}{dt^2} = T^e - \beta\frac{d\theta}{dt}$$

$$\theta = \omega_m t + \gamma_0 + \gamma'(t) \quad , \quad \gamma'(t) << \gamma_0$$

$$-M I_r I_s \sin\gamma_0 - \beta\omega = 0$$

$$\sin\gamma_0 = -\frac{\beta\omega}{M I_r I_s}$$



Pullout when $\left|\sin\gamma_0\right| = 1 \Rightarrow \beta\omega = M I_r I_s$

Hunting transients: $\sin(\gamma_0 + \gamma') \approx \sin\gamma_0 \cos\gamma' + \cos\gamma_0 \sin\gamma' \approx \sin\gamma_0 + \gamma' \cos\gamma_0$

$$J\frac{d^2\gamma'}{dt^2} = -M I_r I_s \cos\gamma_0 \gamma' - \beta\gamma' = -(M I_r I_s \cos\gamma_0 + \beta)\gamma'$$

$$\frac{d^2\gamma'}{dt^2} + \omega_0^2\gamma' = 0 \quad ; \quad \omega_0^2 = \left[M I_r I_s \cos\gamma_0 + \beta\right]/J$$

$$\gamma' = A_1 \sin\omega_0 t + A_2 \cos\omega_0 t$$

Stable if $\omega_0^2 > 0$ (ω_0 real)

Unstable if $\omega_0^2 < 0$ (ω_0 imaginary)